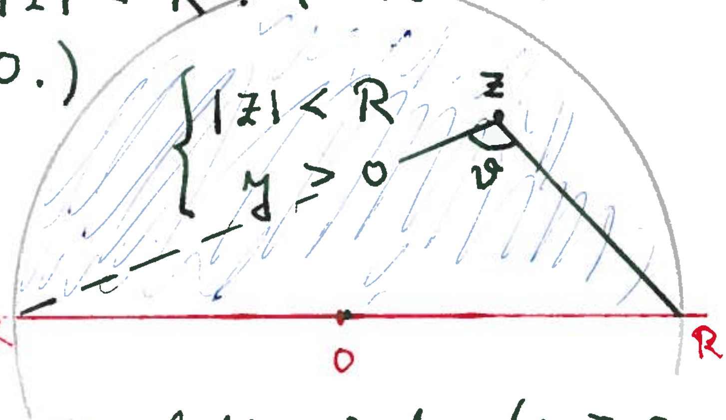
⑦ PHRAGMÈN – LINDELÖF

The function $u(z) = \log|f(z)|$ is subharmonic. The function

$$\omega_R(z) = 2\left(1 - \frac{1}{\pi}\arg\left(\frac{z-R}{z+R}\right)\right)$$
$$= \frac{2}{\pi}\arctan\left(\frac{2Ry}{R^2-|z|^2}\right)$$

$\omega_R(z)$ is a function of the angle ϑ below,

is harmonic when $|z| < R$. (One can calculate $\Delta\omega_R = 0$.)



Now we compare $\omega_R(z)$ and $u(z)$ on the boundary of the half-circle ($y \geq 0$, $|z| = R$)

$$\begin{cases} 1 = \omega_R(z) \geq \dfrac{u(z)}{\max\limits_{\substack{|z|=R\\ y\geq 0}}|u(z)|} & \text{on the arc;} \\[2ex] \omega_R(z) \overset{(>)}{=} 0 \geq u(z) & \text{on the real axis } (y=0,\ -R\leq x\leq R) \end{cases}$$

By the comparison principle

$$\frac{u(z)}{\max\limits_{\substack{|z|=R\\ y\geq 0}}|u(z)|} \leq \omega_R(z) \quad \text{when } |z|\leq R,\ y\geq 0.$$

$$\lim_{R\to\infty}\left(R\,\omega_R(z)\right) = \frac{4}{\pi}y.$$

It follows that $u(z) \leq 0$, if $\limsup\limits_{R\to\infty} \dfrac{\max\limits_{R}|u|}{R} \leq 0$.