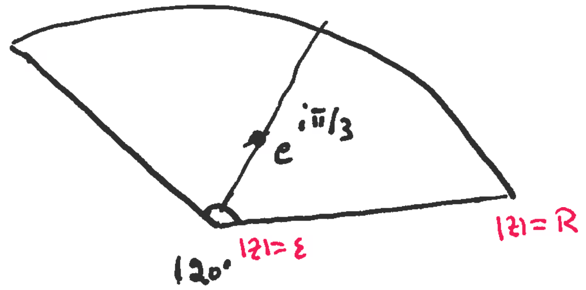
④



$\varepsilon \le |z| \le R$

$$\left(\begin{aligned} z^3+1 &= 0 \\ z^3 &= e^{i\pi + 2ni\pi} \\ z &= e^{i\pi(1+2n)/3} \end{aligned} \right)$$

$$z = re^{i\theta},\quad \log(z) = \log_\varepsilon(r) + i\theta,\quad 0 \le \theta \le \frac{2\pi}{3}.$$

$$\int_\varepsilon^R \frac{\log(x)}{x^3+1}\,dx + e^{i\frac{2\pi}{3}} \int_R \frac{\log(r) + i\frac{2\pi}{3}}{r^3+1}\,dr + \int_{C_\varepsilon} + \int_{C_R}$$

SIMPLE POLE AT $e^{i\pi/3}$.

C_ε: Arc $|z|=\varepsilon$ $\quad$ C_R: Arc $|z|=R$

$$= 2\pi i \cdot \operatorname*{Res}_{z=e^{i\pi/3}} \left\{ \frac{\log(z)}{z^3+1} \right\}$$

$$= 2\pi i \cdot \frac{0 + i\,\pi/3}{3e^{i\pi/3\cdot 2}} = -\frac{2\pi^2}{9} e^{-\frac{2i\pi}{3}}$$

$$\left| \int_{C_\varepsilon} \cdots d\theta \right| \le \int_0^{2\pi/3} \frac{\log(\frac{1}{\varepsilon}) + \frac{2\pi}{3}}{1-\varepsilon^3} \varepsilon\, d\theta \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0+$$

$$\left| \int_{C_R} \cdots d\theta \right| \le \int_0^{2\pi/3} \frac{\log R + \frac{2\pi}{3}}{R^3-1} R\, d\theta \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

Thus [see next page]

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⑥ $$PV \int_0^1 \frac{dx}{1-x^2} = \frac{1}{2} PV \int_{-\infty}^{\infty} \frac{dx}{1-x^2}$$

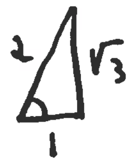
$$= \frac{1}{2} \underbrace{50\% \cdot 2\pi i}_{=\pi i} \left\{ \operatorname*{Res}_{z=1} \frac{1}{1-z^2} + \operatorname*{Res}_{z=-1} \frac{1}{1-z^2} \right\} = 0$$

$$\left(1-e^{\frac{2\pi i}{3}}\right)\int_0^\infty \frac{\log(x)}{x^3+1}\,dx - i\,\frac{2\pi}{3}\int_0^\infty \frac{dx}{x^3+1}\cdot e^{i\frac{2\pi}{3}}$$

$$= -\frac{2\pi^2}{9}\,e^{-\frac{2i\pi}{3}} \qquad \Big|\; e^{-\frac{\pi i}{3}}$$

$$-2i\sin\left(\frac{\pi}{3}\right)\int_0^\infty \frac{\log(x)}{x^3+1}\,dx - i\cdot\frac{2\pi}{3}\,e^{i\pi/3}\int_0^\infty \frac{dx}{x^3+1} = -\frac{2\pi^2}{9}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt3}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}$$



$$\sqrt3\int_0^\infty \frac{\log x}{x^3+1}\,dx + \frac{\pi}{3}\left[\frac12+\frac{\sqrt3}{2}i\right]2\int_0^\infty\frac{dx}{x^3+1} = \frac{2\pi^2}{9}\,i$$

1) $$\frac{\pi}{3}\cdot\sqrt3\int_0^\infty\frac{dx}{x^3+1} = \frac{2\pi^2}{9}, \qquad \int_0^\infty\frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt3}$$

Imag. Part

2) $$\sqrt3\int_0^\infty \frac{\log(x)}{x^3+1}\,dx + \frac{\pi}{3}\cdot\frac{2\pi}{3\sqrt3} = 0$$

Real Part

$$\int_0^\infty \frac{\log(x)}{x^3+1}\,dx = -\frac{2\pi^2}{27}$$

(5) It is most convenient to use the sector $0\le\theta\le\frac{\pi}{n}$, $0\le r\le R\longrightarrow\infty$. There is then only one pole, a simple one at $e^{i\pi/2n}$.