COMPLEX ANALYSIS 2019

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① Study the convergence of

$$\sum_{n=0}^{\infty} \frac{\sin(nz)}{2^n}$$

in the complex plane.

② Show that "KOEBE's Function"

$$w = f(z) = \frac{z}{(1-z)^2}, \qquad |z| < 1,$$

is univalent in the unit disc $|z| < 1$.
Find the image of the unit disc.

③ Determine the limit function of

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

where $\omega = n_1\omega_1 + n_2\omega_2$, as $\omega_2 \longrightarrow \infty$.
For simplicity, $\omega_1 = 1$. (Identify a trigonometric function.)

④ Determine the smallest integer $k \geq 1$ for which the product

$$z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2 + \frac{1}{3}\left(\frac{z}{\omega}\right)^3 + \cdots + \frac{1}{k}\left(\frac{z}{\omega}\right)^k}$$

$\omega = n_1\omega_1 + n_2\omega_2$,

converges absolutely.

$\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) \neq 0.$

(5) Calculate $\zeta(0)$ using

$$\zeta(s) = \frac{e^{-i\pi s}\,\Gamma(1-s)}{2\pi i}\int_\gamma \frac{z^{s-1}}{e^z-1}\,dz$$

"$1+1+1+\cdots = -\frac{1}{2}$"

(6) Derive the duplication formula

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$$

from the addition formula for Weierstrass's $\wp$-function.

(7) Prove the Phragmen-Lindelöf Theorem.
Assume that $f(z)$ is analytic in the upper half-plane $\operatorname{Im}(z) > 0$. Denote

$$M(r) = \sup_{\substack{|z|=r \\ \operatorname{Im}(z)>0}} |f(z)|$$

If

$$\limsup_{z\to x} |f(z)| \leq 1$$

at all points $-\infty < x < \infty$ and

$$\liminf_{r\to\infty}\left(\frac{\log M(r)}{r}\right) \leq 0,$$ then $|f(z)| \leq 1$ in the whole half-plane.