FUNCTIONS OF GROWTH ORDER 1

Weierstraß test for uniform convergence:

1) $|c_n(z)| \le M_n$, when $z \in \Omega$ $(n=1,2,3,\dots)$

free of z

2) $\sum M_n < \infty$

$\Longrightarrow$ $\sum c_n(z)$ converges uniformly in Ω

$$\sum_{n=1}^{\infty} |c_n(z)| < \infty \iff \prod_{n=1}^{\infty} (1+|c_n(z)|)$$

converges (loc. unif.) converges (loc. unif.)

$$\Longrightarrow \prod_{n=1}^{\infty} (1+c_n(z))$$

converges (loc. unif.)

$$\prod_{n=1}^{\infty} \underbrace{\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}}_{1+c_n(z)}$$

In fact, $C=1$ will do.

$$|c_n(z)| = \left|\left(1-\frac{z}{a_n}\right)e^{+\frac{z}{a_n}} - 1\right| \le C\left|\frac{z}{a_n}\right|^2$$

when $\left|\frac{z}{a_n}\right| < 1$

$$1-\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}} = \int_0^{z/a_n} te^t\,dt$$

If $\sum |a_n|^{-2} < \infty$ (which is true for the roots a_n of entire functions of growth order 1 by a Corollary of Jensen's formula), then the product

$$\prod \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

$$= \underbrace{\prod_{|a_n| \le |z|} \text{—}''\text{—}}_{\text{finitely many factors}} \cdot \prod_{|a_n| > |z|} \text{—}''\text{—}$$

is uniformly convergent in each fixed disc $|z| \le L$. (To see this, take the factors with $|a_n| \le L$ separately and consider

$$\prod_{|a_n| > L} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}. \qquad (|z| \le L))$$

CONCLUSION For an entire function of order 1, with zeros $a_1, a_2, \ldots$ the function

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

is analytic

LEMMA An entire function $f(z)$ of finite order β and without zeros is of the form

$$f(z) = e^{g(z)}$$

where $g(z) = \underbrace{c_0 + c_1 z + \cdots + c_m z^m}_{\text{polynomial}}$, $m \leq \beta$.

In fact, $\beta = m$.

Remark The proof shows that if

$$|f(z)| \leq e^{R_j^{\beta+\varepsilon}}, \quad \text{when } |z| = R_j,$$

holds for some sequence $R_j \to \infty$, then $m \leq \beta + \varepsilon$.

Proof: Assume $f(0) = 1$ (can be arranged). Then

$$g(z) = \log f(z) = \overbrace{\log|f(z)|}^{u} + iv$$

$$= \sum_{n=0}^{\infty} (a_n + i b_n) z^n$$

$$0 = u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta})\, d\theta$$

$$u(z) = u(Re^{i\theta}) = \sum_{n=1}^{\infty} a_n R^n \cos(n\theta) - b_n R^n \sin(n\theta)$$

By assumption

$$-\infty < u(Re^{i\theta}) \le C_\varepsilon R^{\beta+\varepsilon} \qquad (\varepsilon > 0)$$

Now the Fourier coefficients are

$$\overset{b_n}{|a_n R^n|} = \left| \frac{1}{\pi} \int_0^{2\pi} u(Re^{i\theta}) \overset{\sin(n\theta)}{\cos(n\theta)}\, d\theta \right|$$

$$\le \frac{1}{\pi} \int_0^{2\pi} |u(Re^{i\theta})|\, d\theta$$

Takes care of $u < 0$.

$$= \frac{1}{\pi} \int_0^{2\pi} \underbrace{(|u(Re^{i\theta})| + u(Re^{i\theta}))}_{\le 2C_\varepsilon R^{\beta+\varepsilon}}\, d\theta$$

$$\le 4 C_\varepsilon R^{\beta+\varepsilon} \implies a_n = 0, \text{ when } n > \beta + \varepsilon.$$

The same for b_n.

The conclusion comes as $R \longrightarrow \infty$. $\square$

THM An entire function of order 1 has the representation

$$f(z) = z^m e^{A+Bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}$$

If $f(0) \ne 0$, $m = 0$. (The product is locally uniformly convergent.)

Proof: The product

$$P(z) = \prod \left(1 - \frac{z}{z_n}\right) e^{+\frac{z}{z_n}}$$

converges absolutely and locally uniformly since $\sum |z_n|^{-2}$ converges. (In fact

$$\sum \frac{1}{|z_n|^{1+\varepsilon}} < \infty$$

when $\varepsilon > 0$.) Assume $f(0) \neq 0$. Now

$$F(z) = \frac{f(z)}{P(z)} = e^{g(z)}$$

since $F(z)$ has no zeros. It suffices to show that

$$|F(z)| \leq C_\varepsilon e^{R_j^{1+\varepsilon}}, \quad |z| = R_j$$

for some sequence $R_j \longrightarrow \infty$, in order to conclude that $g(z) = A + Bz$.

~~To this end, choose R_j satisfying~~

$$\cancel{\left| R_j - |z_n| \right| > |z_n|^{-2}}$$

~~for all $n = 1, 2, 3, \ldots$ simultaneously.~~

[Landau's argument avoids this traditional complication!]

Recall

$$|f(z)| \leq C e^{B|z|^\beta} \quad \text{for all } \beta > 1.$$

$$1 = \limsup_{n\to\infty} \frac{\log\log(M(n))}{\log n}. \quad \text{Growth order} = 1.$$

To this end, write

$$P(z) = \underbrace{\prod_{|z_n|\le 2R} \left(1-\frac{z}{z_n}\right) e^{z/z_n}}_{P_1(z)} \cdot \underbrace{\prod_{|z_n|>2R} \left(1-\frac{z}{z_n}\right) e^{z/z_n}}_{P_2(z)}$$

and abbreviate

$$E_1(w) = (1-w)e^{w}.$$

① $$\boxed{\min_{|z|=R} |P_2(z)| \ge e^{-C_1 R^{\beta}}} \qquad (1<\beta<2)$$

Now $\left|\frac{z}{z_n}\right| \le 2^{-1}$ in each factor of $P_2(z)$. Use

$|E_1(w)| \ge e^{-2|w|^2}$ for $|w| \le \frac{1}{2}$. Thus

$$|P_2(z)| = \prod_{|z_n|>2R} \left|E_1\left(\frac{z}{z_n}\right)\right| \ge \prod_{|z_n|>2R} e^{-2\left|\frac{z}{z_n}\right|^2}$$

$$\ge \prod_{|z_n|>2R} e^{-2\left|\frac{z_n}{z_n}\right|^{\beta}} = e^{-2\sum_n \left|\frac{z}{z_n}\right|^{\beta}}$$

$$\ge e^{-C_1 R^{\beta}}$$

(Recall $\sum_{n=1}^{\infty} |z_n|^{-\beta} < \infty$.)

② Estimate of $P_1(z)$ when $|z| = 4R$. (This clever device is due to Landau.)

$$\boxed{\min_{|z|=4R} |P_1(z)| \ge e^{-C_2 R^{\beta}}.}$$

Now $|z/z_n| \ge 2$ for all factors in $P_1(z)$. Use

$$|E(w)| \geq |1-w|\, e^{-|w|} \geq \underbrace{(|w|-1)}_{\geq 2-1=1} e^{-|w|}$$

$$\geq e^{-|w|} \quad \text{since } |w| \geq 2.$$

$$\left|P_2\left(\frac{z}{z_n}\right)\right| \geq \prod_{|z_n|\leq 2R} e^{-\frac{|z|}{|z_n|}} = e^{-\sum_n \left|\frac{z}{z_n}\right|}$$

$$\geq e^{-\sum_n \left|\frac{z}{z_n}\right|^\beta} \geq e^{-C_2 R^\beta} \qquad (\beta > 1)$$

③ Final estimate on $|z| = R$.

$$\max_{|z|=R} \left|\frac{f(z)}{P(z)}\right| \leq \max_{|z|=R} \left|\frac{f(z)}{P_1(z)}\right| \max_{|z|=R} \left|\frac{1}{P_2(z)}\right|$$

$$\overset{*)}{\leq} \max_{|z|=4R} \left|\frac{f(z)}{P_1(z)}\right| \max_{|z|=R} \left|\frac{1}{P_2(z)}\right|$$

*) $\frac{f(z)}{P_1(z)}$ is entire, the zeros cancel. Use Maximum Principle!

$$\leq C e^{B(4R)^\beta} e^{+C_2 R^\beta} e^{C_1 R^\beta} = C e^{C_3 R^\beta}$$

where $\beta > 1$ is arbitrarily close to 1.

This means that

$$e^{\mathrm{Re}\{g(z)\}} = |e^{g(z)}| = \left|\frac{f(z)}{P(z)}\right| \leq C e^{C_3 |z|^\beta}$$

as $z \to \infty$. Thus $g(z)$ is a polynomial of order $\leq \beta$ (Lemma), i.e., $g(z) = az + b$.

④ If $m \geq 1$, use $z^{-m} f(z)$ in the place of $f(z)$ above.

□