$$e^z = \sum_{n=0}^{\infty} z^n/n! \neq 0.$$

Let $R >> 1$. Show that there exists an index N such that when $n > N$ all roots of the polynomial

$$1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

lie outside the disk $B(0,R)$.

Proof:

$$\underbrace{1 + z + \cdots + \frac{z^n}{n!}}_{= P_n(z)} + R_n(z) = e^z$$

Fix N so that

$$\max_{|z|=R} |R_n(z)| < \frac{e^{-R}}{2}.$$

When $|z| = R$

$$|P_n(z)| = |e^z - R_n(z)| \geq |e^z| - |R_n(z)|$$

$$\geq e^{-R} - |R_n(z)| > \frac{1}{2} e^{-R}$$

By Rouche's Theorem $P_n(z)$ and $P_n(z) + R_n(z) = e^z$ have the same number of zeros in $|z| < R$. But $e^z \neq 0$. The claim follows. □

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$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{(1+i\omega)^2}, \qquad f(x) = ?$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(1+i\omega)^2}\, d\omega$$

We write

$$f(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iax}}{(1+ix)^2}\, dx$$

and consider

$$\oint \frac{e^{iaz}}{(1+iz)^2}\, dz .$$

On a circle of radius R, $|z| = R > 1$.

$$\left| \frac{e^{iaz}}{(1+iz)^2} \right| \leq \frac{e^{-ay}}{R^2 - 1}$$

1°) $\underline{a > 0}$ Half circle $|z| = R$, $\underline{y > 0}$, in the upper half plane $y > 0$

$$\left| \int_{D_R^+} \frac{e^{-az}}{(1+iz)^2}\, dz \right| \leq \frac{\pi R \cdot 1}{R^2 - 1} \longrightarrow 0.$$

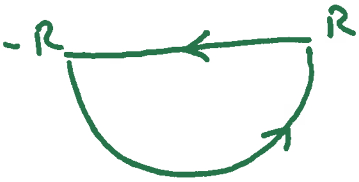
$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+ix)^2}\, dx = 2\pi i \operatorname*{Res}_{z=i} \left\{ \frac{e^{iaz}}{(1+iz)^2} \right\} = 2\pi i \cdot \frac{e^{-a} a}{i} = 2\pi e^{-a} a$$

$$\frac{e^{-a} \cdot e^{a(iz+1)}}{(iz+1)^2} = e^{-a} \left[\frac{1}{(iz+1)^2} + \frac{a}{\underbrace{iz+1}_{i(z-i)}} + \frac{a^2}{2!} + \cdots \right]$$

This term yields the residue.

2°) $\underline{a < 0}$ Half circle in the lower half plane $\underline{y < 0}$

$\underline{\underline{ay > 0}}$



Again the integral on D_R^- approaches zero.

No singularities now!

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(1+ix)^2}\,dx = 0$$

Conclusion

$$f(x) = \begin{cases} 0, & x < 0 \\ xe^{-x}, & x \geq 0 \end{cases}$$