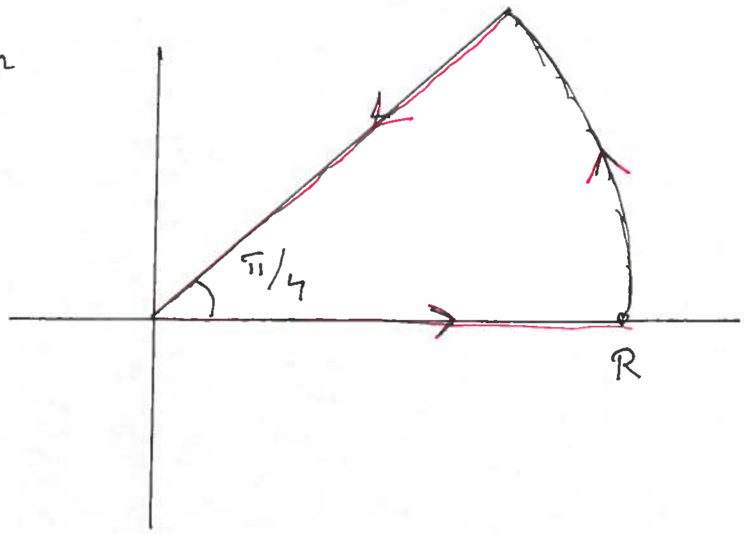
① $$\int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}, \quad \int_{-\infty}^{\infty} \cos(x^2)\,dx = \sqrt{\frac{\pi}{2}}$$

To see this, calculate

$$\oint e^{iz^2}\,dz = 0$$

along the sector



$$\int_0^R e^{ix^2}\,dx + \int_0^{\pi/4} e^{i(Re^{i\theta})^2} Re^{i\theta} i\,d\theta$$

$$+ \underbrace{\int_R^0 e^{-r^2} e^{i\frac{\pi}{4}}\,dr}_{= -\frac{1+i}{\sqrt{2}}\frac{\sqrt{\pi}}{2}} = 0$$

Take the real part.

$$\left|\int_0^{\pi/4} e^{iR^2e^{2i\theta}} Re^{i\theta} i\,d\theta\right| \le R\int_0^{\pi/4} e^{-R^2\sin(2\theta)}\,d\theta \longrightarrow 0$$

as $R \to \infty$.

$\sin(2\theta) \ge \frac{2}{\pi}\cdot 2\theta$

(2) $z \to z^3$ maps the sector into the upper half-plane. Then

$$w = \frac{z^3 - i}{z^3 + i}$$

will do.

(3a) $R = \frac{1}{3}$

(3b) Split $\sum_n = \sum_{\substack{\text{odd}\\ n}} + \sum_{\substack{\text{even}\\ n}}$ to get two geometric series.

$$f(z) = \frac{1}{1 - 9z^2} + \frac{z}{1 - z}.$$

(4) $e^{-\pi/2} e^{-2n\pi} \quad (n = 0, \pm 1, \pm 2, \ldots)$

(5) $$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \qquad \begin{matrix} x = nt \\ dx = n\,dt \end{matrix}$$

$$= \int_0^\infty e^{-nt} t^{s-1} n^s\, dt$$

$$\Gamma(s) n^{-s} = \int_0^\infty e^{-nt} t^{s-1}\, dt$$

Geometric Series

$$\Gamma(s)\zeta(s) = \sum \int_0^\infty \cdots dt = \int_0^\infty \left(\sum_{n=1}^\infty e^{-nt}\right) t^{s-1}\, dt$$

$$= \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt \quad \text{valid for } \underline{\operatorname{Re}(s) > 1}$$

[Change s to z.]

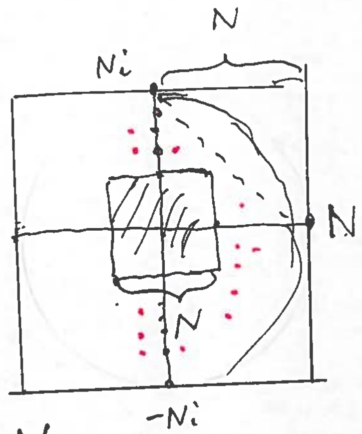
⑥

We claim that $f(z) \equiv 0$. If not, then

$$f(z) = c z^m + d z^{m+1} + \cdots \quad (m \geq 0)$$

and Jensen's formula takes the form

$$\left\{\begin{aligned} &\sum_{0<|z_j|\leq R} \log\left(\frac{R}{|z_j|}\right) + \log|c| + m \log R \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} \log\left(\left|f(Re^{i\theta})\right|\right) d\theta \end{aligned}\right.$$



$$\leq \log(2019) + 100R.$$

Here $f(z_j) = 0$. Now for $R = N$

$$\sum_{0<|z_j|\leq R} \log\left(\frac{R}{|z_j|}\right) \geq \sum_{\frac{N}{2}\leq|z_j|\leq R} \log\frac{N}{|z_j|} \geq \underbrace{\left[\frac{(2N)^2}{2} - N^2\right]}_{\substack{\text{A rough} \\ \text{estimate of} \\ \text{the number of} \\ \text{zeros given.}}} \log\left(\frac{N}{\frac{N}{2}}\right)$$

$$= N^2 \log 2$$

As $N \to \infty$, $N^2 \log 2 \lesssim 100N$ yields a contradiction. Hence $f \equiv 0$. ∎

NOTICE: $f(m+ni) = 0$ is needed.

$$m, n = \pm 1, \pm 2, \pm 3, \ldots$$