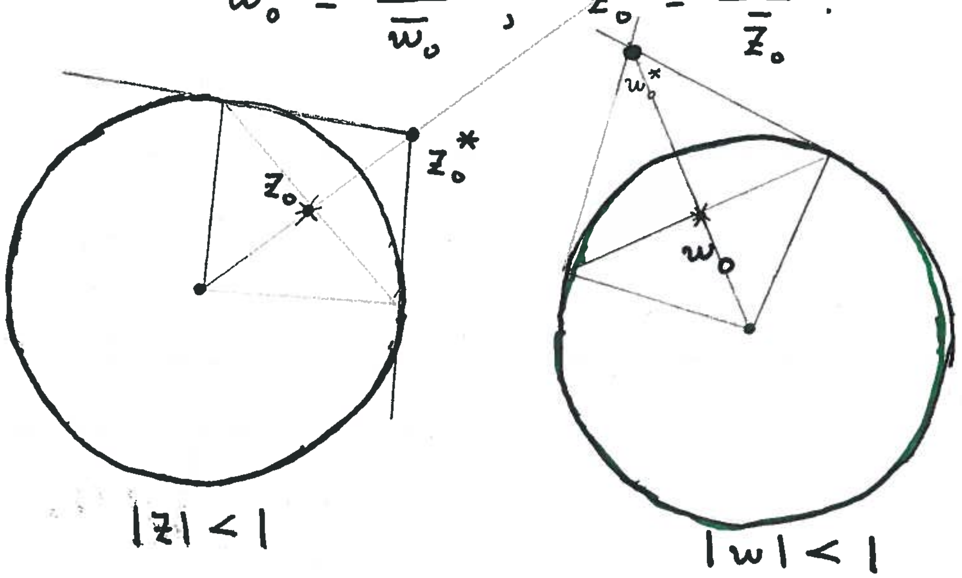
The Conformal Mappings of the Unit Disc onto Itself

As we will later[*)] see such a mapping must be a Möbius transformation. Let w_0 and z_0 correspond to each other, $|z_0| < 1$, $|w_0| < 1$. So do the symmetric points

$$w_0^* = \frac{1}{\overline{w}_0}, \quad z_0^* = \frac{1}{\overline{z}_0}.$$



$w = w(z)$

$w_0 = w(z_0)$

The data yield the transformation

$$\frac{w - w_0}{w - w_0^*} = K \frac{z - z_0}{z - z_0^*},$$

$$\frac{w - w_0}{1 - \overline{w}_0 w} = \eta \frac{z - z_0}{1 - \overline{z}_0 z} \qquad \eta = \frac{K \overline{z}_0}{\overline{w}_0}$$

Taking absolute signs we obtain that η is unimodular: $|\eta| = 1$. More precisely,

*) Schwarz lemma.

the circles $|w| = 1$ and $|z| = 1$ correspond, and

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = 1, \quad \text{when } |z| = 1.$$

The same for w. Indeed,

$$\left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = |\bar{z}| \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| = \left| \frac{\overbrace{z\bar{z}}^{=1} - z_0 \bar{z}}{1 - \bar{z}_0 z} \right|$$

$$= \left| \frac{\overline{1 - \bar{z}_0 z}}{1 - \bar{z}_0 z} \right| = 1, \quad \text{when } |z| = 1.$$

Thus

$$\boxed{\frac{w - w_0}{1 - \bar{w}_0 w} = e^{i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z} \qquad (\alpha \text{ real})}$$

$\eta = e^{i\alpha}$

is the general form of a conformal map between the discs $|z| \leq 1$ and $|w| \leq 1$. Here $z = z_0$ if and only if $w = w_0$. If z_0 is selected so that its image is $w_0 = 0$, the formula reduces to

$$w = e^{i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z}.$$

$w(z_0) = 0$

If the origin remains fixed ($w_0 = z_0 = 0$), we have merely the rotation $w = e^{i\alpha} z$. For the calculation of the <u>fixed points</u>, the alternative formula

$$w = \frac{az + b}{\bar{b} z + \bar{a}} \qquad (|b| < |a|)$$

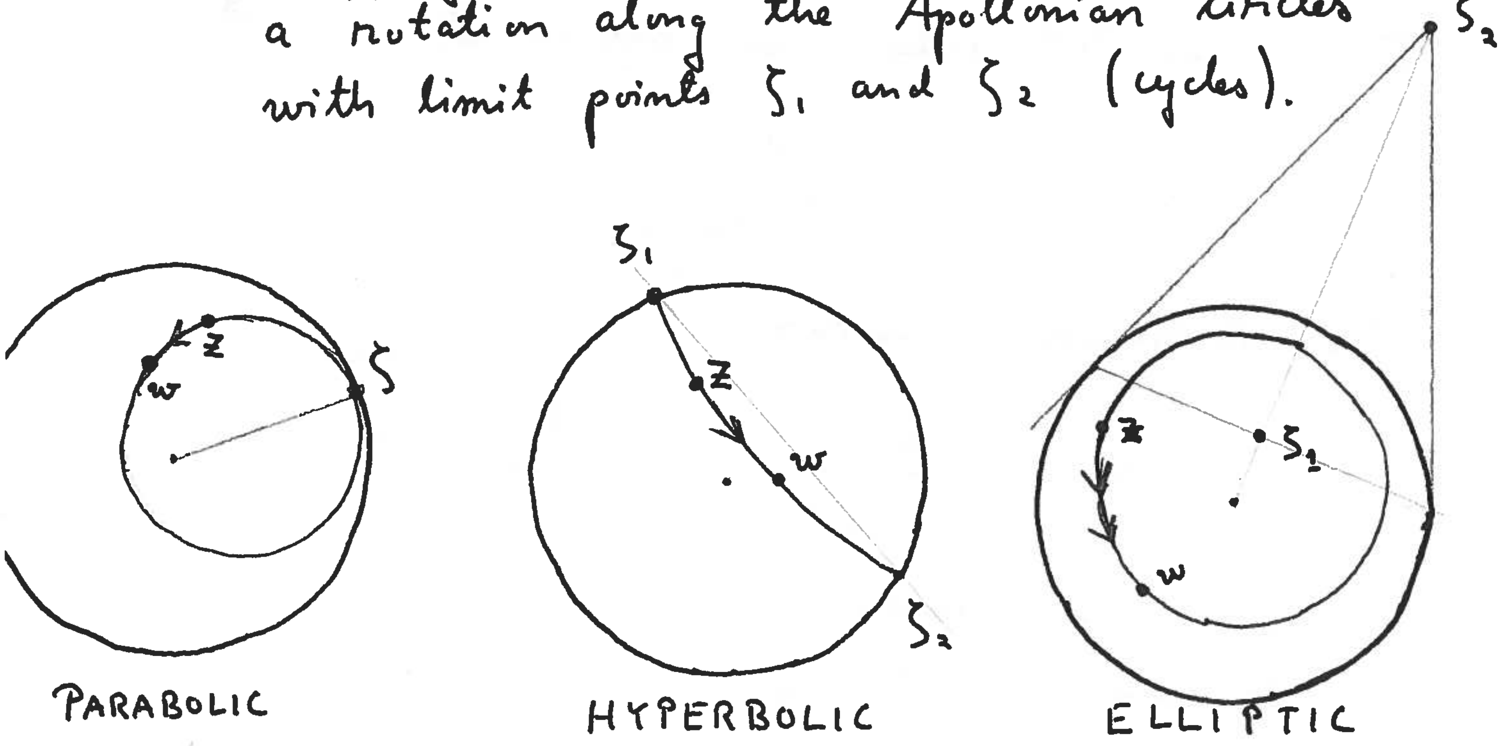
is convenient.

There are three possibilities.

I The fixed points coincide to a point ζ on the boundary circle. The transformation is called <u>parabolic</u> and the mapping is a flow along the circles tangent to the boundary circle at ζ (horocycles).

II The fixed points ζ_1 and ζ_2 are different and lie on the boundary circle. The transformation is called <u>hyperbolic</u>. The flow is along the Steiner circles through ζ_1 and ζ_2 (hypercycles)

III The fixed points ζ_1 and ζ_2 are symmetric ($\zeta_1 \bar{\zeta}_2 = 1$), one is inside the circle. The mapping is called <u>elliptic</u>. The flow is like a rotation along the Apollonian circles with limit points ζ_1 and ζ_2 (cycles).



All other situations are excluded.

In each case the unit circle itself is one of the Steiner (Apollonius) circles.

Noneuclidian Distance

Write the mapping as

$$(*) \qquad \frac{w - w_0}{z - z_0} = e^{i\alpha} \frac{1 - \overline{w}_0 w}{1 - \overline{z}_0 z}$$

and take the limit as $z \to z_0$. We have

$$\left. \frac{dw}{dz} \right|_{z = z_0} = \lim_{z \to z_0} \frac{w(z) - w(z_0)}{z - z_0} .$$

Hence

$$\left. \frac{dw}{dz} \right|_{z = z_0} = e^{i\alpha} \frac{1 - |w_0|^2}{1 - |z_0|^2}$$

Taking absolute values we arrive at

$$\frac{|dw|}{1 - |w|^2} = \frac{|dz|}{1 - |z|^2}$$

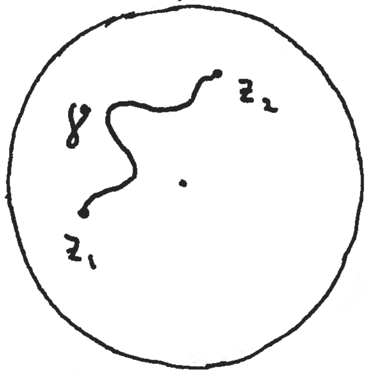
The differential expression

$$d\sigma = \frac{|dz|}{1 - |z|^2} = \frac{\sqrt{(dx)^2 + (dy)^2}}{1 - x^2 - y^2}$$

is <u>invariant</u> under conformal self-mappings of the unit disk! We define, following

Poincaré (1854–1912), the hyperbolic length of the curve γ as

$$\int_\gamma d\delta = \int_{z_1}^{z_2} \frac{|dz|}{1-|z|^2}$$



The disc $|z|<1$

The shortest hyperbolic length between z_1 and z_2 is measured along a geodesic curve. It turns out that the geodesics are the circles orthogonal to the unit circle. The hyperbolic distance between z_1 and z_2 is the aforementioned shortest length, measured along the corresponding orthogonal circle (also the diameters are regarded as such circles). One gets the answer

$$\frac{1}{2}\log\left(\frac{1+\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}\right),$$

(REMARK: Sometimes the factor 2 is included in the definition.)

cf. Ahlfors, 4 § 3 Exc. 7. In particular, the hyperbolic distance from 0 to z is

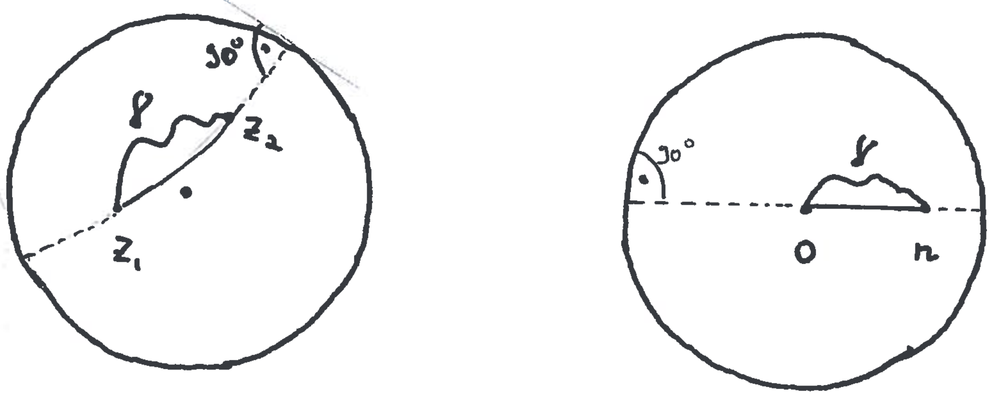
$$\frac{1}{2}\log\frac{1+|z|}{1-|z|}$$

and it approaches ∞ as z approaches the circle $|z|=1$. Hence the boundary is at infinite distance, as it were!

For the calculation of all this, map z_1 to $w_1 = 0$ using

$$w = e^{i\alpha}\,\frac{z - z_1}{1 - \bar{z}_1 z}.$$

This maps z_2 to a point w_2. Rotate the circle so that w_2 is on the real axis, $w_2 = re^{i0}$, $0 < r < 1$ (it amounts to the same thing to choose α in a suitable way).



Because of the invariance

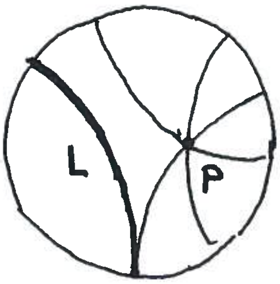
$$\int_\gamma ds = \int_{w_1=0}^{w_2} \frac{|dw|}{1-|w|^2} \geq \int_0^r \frac{|du|}{1-u^2} = \int_0^r \frac{du}{1-u^2}$$

$$= \frac{1}{2}\log\frac{1+r}{1-r}, \quad \text{where } r = |w_2| = \frac{|z_2 - z_1|}{|1-\bar{z}_1 z_2|}$$

since $|dw| = \sqrt{(du)^2 + (dv)^2} \geq |du|$ and $1 - |w|^2 = 1 - u^2 - v^2 \leq 1 - u^2$. Equality requires $v \equiv 0$. Thus γ is a geodesic if and only if its image is the segment $[0, r]$. The corresponding diameter is the image of an orthogonal circle.

This concludes our proof of the fact that the geodesics are the orthogonal circles.

An interesting application is Poincaré's model of the noneuclidian plane (the geometry of Bólyai – Lobatchevskij; hyperbolic plane). The <u>lines</u> are the aforementioned geodesics (the shortest distance is measured along a "line") in the unit circle. The axioms of Euclidean geometry are valid with one noteworthy exception: the parallel axiom.



"Lines" that do not intersect are "parallel". Through the point P one can draw infinitely many "lines" parallel to L.

If the radius of the circle "is ∞", we have the Euclidean situation.

The element of area is

$$d\omega = \frac{dx\,dy}{(1-|z|^2)^2}$$

<u>Remark</u>: Also possible in n-dimensions. Similar formulas for $(x_1, x_2, \dots, x_n)$

R. Nevanlinna, V. Paatero: Introduction to Complex Analysis (Chapter 3)