COMPLEX ANALYSIS / 11

1801–1892

① The AIRY function is defined as

$$Ai(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{x^3}{3} + sx\right)}\,dx\,, \qquad s \in \mathbb{R}.$$

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−2,3
−4,1
−5,5
−6,7
−7,9
−9,0
−10,0
−11,0

Show that the integral converges. Then move the "contour" of integration to $z = x + i\delta$, $\delta > 0$.

Show that

$$Ai''(s) = s\,Ai(s)$$

(This simple differential equation explains the ubiquity of the Airy function.)

$$Ai(0) = \frac{1}{3^{1/6}\,\Gamma\left(\frac{2}{3}\right)}$$

② Use the Euler product

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

to sum the series $\sum n^{-2}$ (this is $\zeta(2)$).

③ Let the function $w(z)$ be analytic in the ring $r_1 \le |z| \le r_2$ and denote

$$M(r) = \max_{|z|=r} |w(z)|\,, \qquad r_1 \le r \le r_2$$

Show that

$$\log M(r) \le \frac{\log\left(\frac{r_2}{r}\right)}{\log\left(\frac{r_2}{r_1}\right)} \log M(r_1) + \frac{\log\left(\frac{r}{r_1}\right)}{\log\left(\frac{r_2}{r_1}\right)} \log M(r_2)$$

when $r_1 \le r \le r_2$. [HADAMARD's three circles thm].

④ Show that the function

$$w = \int_0^z \frac{dz}{\sqrt{z(1-z^2)}}$$

maps the half-plane $\operatorname{Im}(z) > 0$ onto a square.
Show that the sides of the square have the length

$$a = \frac{1}{2}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt = \frac{1}{2\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right).$$

⑤

Construct an entire function $f(z)$ that has the zeros

$$n e^{2\pi i k/n} \quad (k = 0, 1, 2, \ldots, n-1; \; n = 1, 2, 3, \ldots)$$

Hint Weierstrass' product, Hadamard's theorem. Recall that

$$\prod_{\nu=1}^{\infty}\left(1-\frac{z}{z_\nu}\right) e^{\frac{z}{z_\nu} + \frac{1}{2}\left(\frac{z}{z_\nu}\right)^2 + \cdots + \frac{1}{k}\left(\frac{z}{z_\nu}\right)^k}$$

converges locally uniformly, if

$$\sum_{\nu=1}^{\infty}\left|\frac{z}{z_\nu}\right|^{k+1} < \infty.$$