POSITIVE TERMS

$$\sum_{n=1}^{\infty} a_n \,, \quad a_n \geq 0$$

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$S_1 \leq S_2 \leq S_3 \leq \ldots$$ INCREASING!

The series with positive terms converges $\Longleftrightarrow$ The partial sums are bounded:

$$S_n < M \quad (n = 1, 2, 3, \ldots)$$

Indep. of n!

It follows:

"COMPARISON"

$$0 \leq a_n \leq b_n$$

MAJORANT

$$\sum b_n \text{ conv.} \Longrightarrow \sum a_n \text{ conv.}$$

$$\sum a_n = \infty \Longrightarrow \sum b_n = \infty$$

MINORANT

ROOT TEST, RATIO TEST, INTEGRAL TEST

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \qquad \lim_{n\to\infty} \sqrt[n]{a_n} \qquad \sum_{n=1}^{\infty} f(n) \approx \int_1^{\infty} f(x)\,dx$$

ABSOLUTE CONV.

$$\sum |a_n| < \infty \Rightarrow \sum a_n \text{ converges}$$

PROOF

$$a_n = \frac{|a_n| + a_n}{2} - \frac{|a_n| - a_n}{2}$$

$$0 \le \frac{|a_n| \pm a_n}{2} \le |a_n|$$

MAJORANT PRINCIPLE $\Rightarrow \sum \frac{|a_n| \pm a_n}{2}$ both converge

So does their difference $\sum a_n$ ■

COMPLEX TERMS

$$\sum z_n = \sum x_n + i \sum y_n$$

$\sum z_n$ conv $\Leftrightarrow$ $\sum x_n$ & $\sum y_n$ converge

$$\sum |z_n| < \infty \Rightarrow \sum z_n \text{ converges}$$

$$\frac{1}{\pi} = \frac{\sqrt{8}}{99^2} \sum_{n=0}^{\infty} \frac{(4n)!\,(1103 + 26390n)}{(4 \cdot 99)^{4n} (n!)^4}$$

RAMANUJAN

MULTIPLICATION

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + a_0 b_3 x^3 + \cdots$$

$$a_1 b_0 x + a_1 b_1 x^2 + a_1 b_2 x^3 + \cdots$$

$$a_2 b_0 x^2 + a_2 b_1 x^3 + a_2 b_2 x^4 + \cdots$$

$$a_3 b_0 x^3 + a_3 b_1 x^4$$

$$\cdots$$

I II III

$$a_0 b_0 + \overset{\text{I}}{(a_0 b_1 + a_1 b_0)}\, x$$

$$+ \overset{\text{II}}{(a_0 b_2 + a_1 b_1 + a_2 b_0)}\, x^2$$

$$+ \overset{\text{III}}{(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)}\, x^3$$

$$+ \cdots$$

$$\sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0)\, x^n$$

<u>Ex</u> $\left(1 + z + z^2 + \cdots\right)^2$

$$= 1 + 2z + 3z^2 + 4z^3 + \cdots$$

$$\sum_{n=1}^{\infty} n z^{n-1} = \left(\frac{1}{1-z}\right)^2, \quad |z| < 1.$$

THE POISSON KERNEL

$$1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1-z} \qquad \text{GEOM. SERIES}$$

Take $z = re^{i\theta}$, $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{1}{1 - re^{i\theta}} = \frac{1 - re^{-i\theta}}{|1 - re^{i\theta}|^2}$$

$$= \frac{1 - r\cos(\theta) + i r \sin(\theta)}{1 - 2r\cos(\theta) + r^2}$$

Take the real part, subtract $1/2$. Thus

$$\frac{1}{2} + r\cos(\theta) + r^2\cos(2\theta) + r^3\cos(3\theta) + \cdots$$

$$= \frac{1 - r^2}{2(1 - 2r\cos(\theta) + r^2)} \qquad (0 \le r < 1)$$