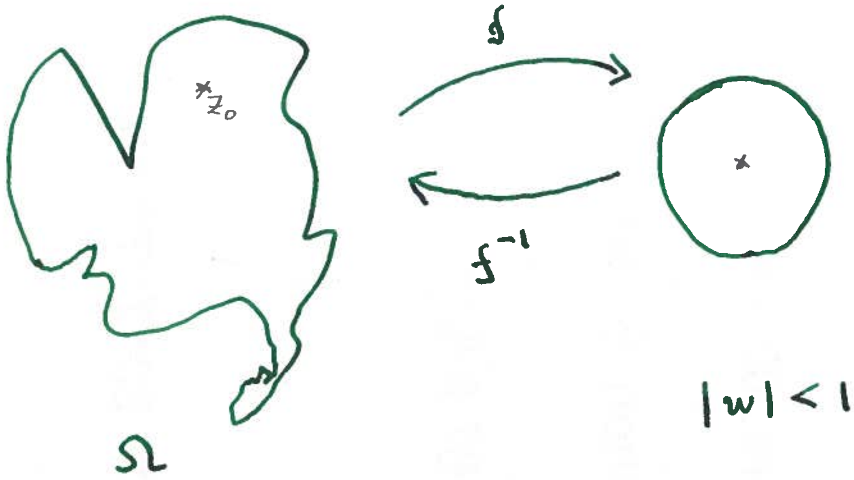
Peter Lindqvist 2018

BOUNDARY BEHAVIOUR OF THE RIEMANN MAPPING



$$f(z_0) = 0, \quad f'(z_0) > 0$$

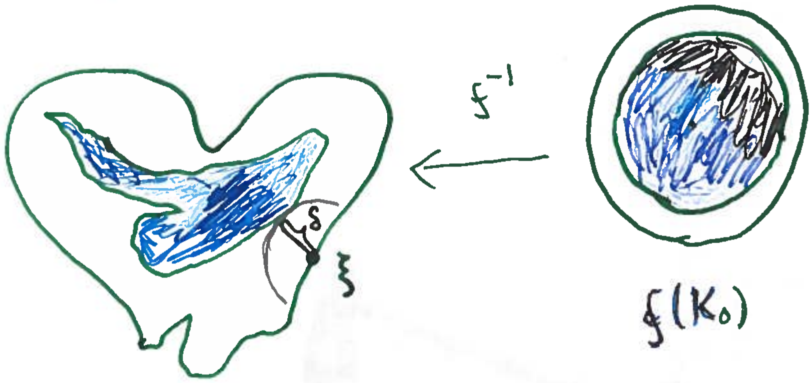
The <u>interior</u> of the simply connected domain Ω, $\Omega \neq \mathbb{C}$, is mapped conformally onto the unit disk $|w| < 1$. It is not always possible to extend f continuously to the boundary $\partial\Omega$ so that the mapping from the closure $\overline{\Omega}$ be continuous. (This is possible by an advanced theorem of Carathéodory, if Ω is bounded by a Jordan curve). In any case, we have

THEOREM If $z_1, z_2, \ldots$ are points in Ω such that

$$\lim_{n \to \infty} z_n = \zeta \in \partial\Omega,$$

then $$\lim_{n\to\infty} |f(z_n)| = 1 .$$

<u>Proof</u>: Let $0 < r_0 < 1$ and suppose that $K_0 \subset \Omega$ is the set that is mapped onto the disk $|w| \le r_0$

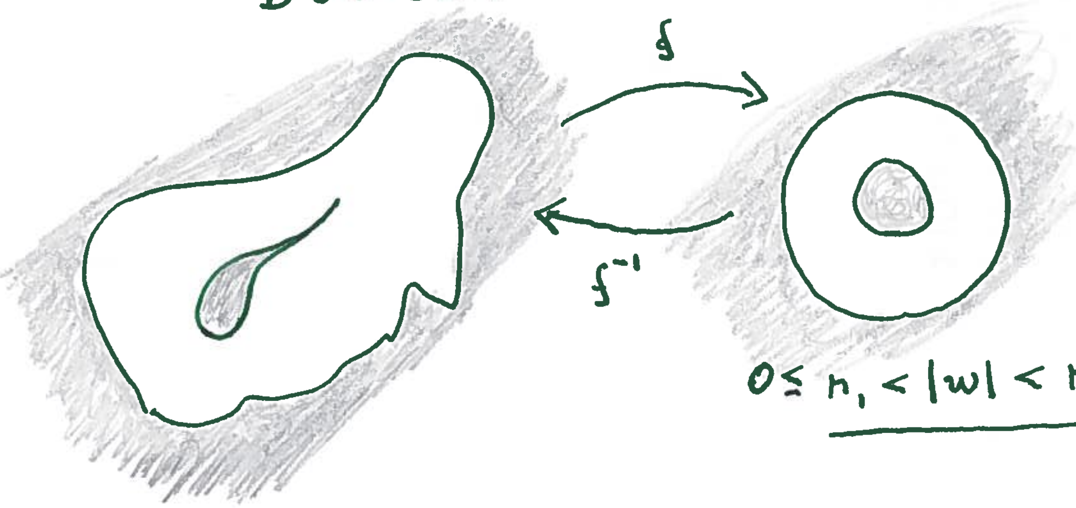


$K_0 \subset\subset \Omega$

Let $\delta = \min\limits_{z \in K_0} |\xi - z|$. Now $|\xi - z_n| < \delta$ when $n > N_0$ (= some index). Thus $z_n \notin K_0$ when $n > N_0$. Hence $|f(z_n)| > r_0$ when $n > N_0$. The result follows.

<u>Remark</u>: Although $|f(z_n)| \longrightarrow 1$, it does not follow that $f(z_n)$ converges. (The points $w_n = f(z_n)$ can ~~spiral~~ follow a spiral, for example.)

DOUBLY CONNECTED DOMAINS



$0 \le r_1 < \underline{|w| < r_2} \le \infty$

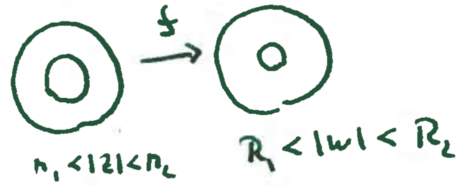
A doubly connected domain can always be conformally mapped onto *some* annulus $r_1 < |w| < r_2$. The ratio r_2/r_1 is decisive.

THEOREM The annuli

$$r_1 < |z| < r_2 \quad \text{and} \quad R_1 < |w| < R_2$$

can be conformally mapped onto each other if and only if

$$\frac{r_2}{r_1} = \frac{R_2}{R_1}.$$



Proof: Assume that $w = f(z)$ is the desired conformal mapping. As $|z| \longrightarrow r_1$ or r_2, we have $|f(z)| \longrightarrow R_1$ or R_2. We may assume that r_1 corresponds to R_1.

Step I $\quad \log(f(z)) = \underbrace{\log|f(z)|}_{\text{Single valued}} + i\arg(f(z))$

Here $h(z) = \log|f(z)|$ is harmonic when $r_1 < |z| < r_2$ and it has the boundary values $\log R_1$ resp. $\log R_2$.

Step II The function

$$k(z) = \frac{\log\left(\frac{|z|}{r_1}\right)}{\log\left(\frac{r_2}{r_1}\right)}\log R_2 + \frac{\log\left(\frac{r_2}{|z|}\right)}{\log\left(\frac{r_2}{r_1}\right)}\log R_1$$

is harmonic in $r_1 < |z| < r_2$. It has the same boundary values on the circles as $\log|f(z)|$. By uniqueness, the functions coincide:

$$\log|f(z)| = \frac{\log\left(\frac{|z|}{r_1}\right)}{\log\left(\frac{r_2}{r_1}\right)} \log R_2 + \frac{\log\left(\frac{r_2}{|z|}\right)}{\log\left(\frac{r_2}{r_1}\right)} \log R_1 \qquad (r_1 < |z| < r_2)$$

Step III Hence,

$$|f(z)| = \left(\frac{|z|}{r_1}\right)^{\log R_2 / \log\left(\frac{r_2}{r_1}\right)} \left(\frac{r_2}{|z|}\right)^{\log R_1 / \log\left(\frac{r_2}{r_1}\right)}$$

$$|f(z)| = C|z|^{\alpha}, \quad \alpha = \frac{\log\left(\frac{R_2}{R_1}\right)}{\log\left(\frac{r_2}{r_1}\right)}$$

Step IV $z^{\alpha} = r^{\alpha} e^{i\alpha\theta}$ is locally defined in the annulus (make a slit from 0 to ∞). Write

$$\left|\frac{f(z)}{z^{\alpha}}\right| = C$$

It follows that $\dfrac{f(z)}{z^{\alpha}} = \text{Constant}$

and

$$f(z) = c\, z^{\alpha}$$

Continuity requires that α is an integer. Injectivity requires that $\alpha = \pm 1$, i.e.

$$\frac{R_2}{R_1} = \frac{r_2}{r_1} \quad \text{or} \quad \frac{R_2}{R_1} = \frac{r_1}{r_2}.$$

This proves the theorem, since the mapping is easy to determine in this case. $\square$