THE FUNCTIONAL EQUATION FOR RIEMANN'S ZETA FUNCTION

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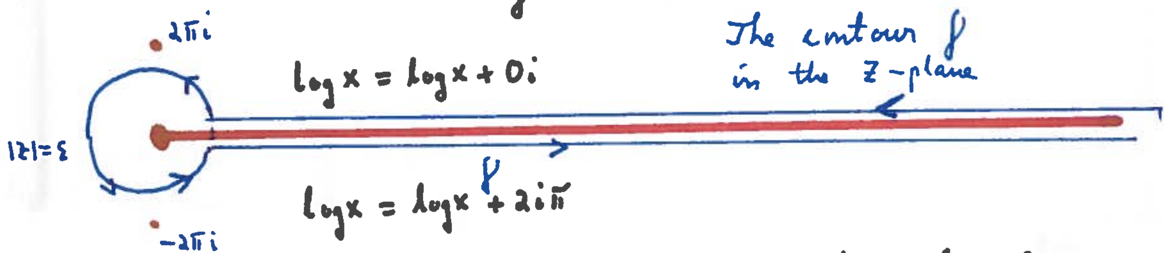
I $$\boxed{\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p \frac{1}{1-p^{-s}}, \quad \sigma = \Re(s) > 1}$$ EULER

II $$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\,dx \qquad (\sigma > 1)$$ RIEMANN

III $$\zeta(s) = \frac{e^{-\pi i s}\,\Gamma(1-s)}{2\pi i} \int_{\gamma} \frac{z^{s-1}}{e^z - 1}\,dz$$

(Independent of $\varepsilon > 0$, small enough.)



The contour γ in the z-plane

Now the integral above is an analytic function of s for all complex s. (We have to keep $0 < \varepsilon < 2\pi$.) Thus III extends $\zeta(s)$ to the whole complex plane. It is a meromorphic function with a simple pole at $s = 1$, because $(1-s)\Gamma(1-s) \longrightarrow 1$ as $s \longrightarrow 1$. (In fact,

$$\zeta(s) = \frac{1}{s-1} + \overset{0{,}5772\ldots}{\gamma} + \gamma_1(s-1) + \gamma_2(s-1)^2 + \cdots,$$

where the coefficients are known.)

Replace γ with the contour γ_n containing a circle of radius $(2n+1)\pi$. The residues at $\pm 2\pi i, \pm 4\pi i, \ldots, \pm 2n\pi i$ have to be taken into account. The Residue Theorem yields

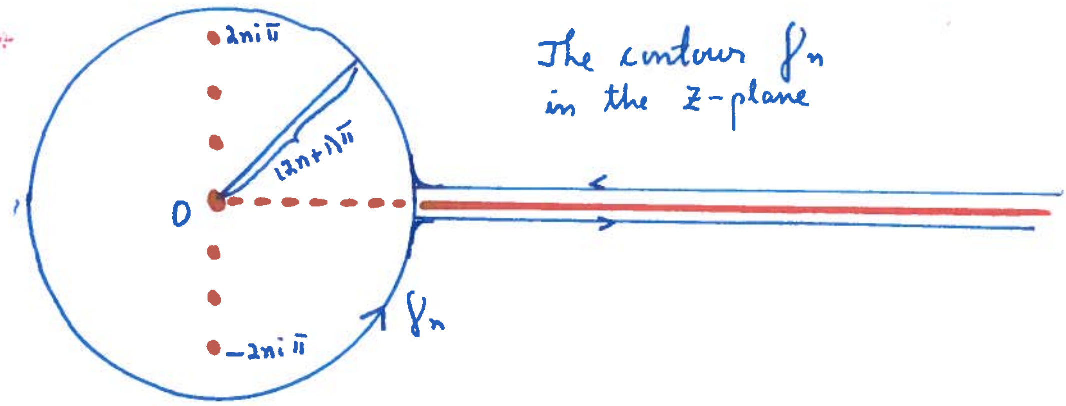
$$\zeta(s) = \frac{e^{-\pi i s}}{2\pi i}\Gamma(1-s)\int_{\gamma_n}\frac{z^{s-1}}{e^z-1}\,dz$$

$$+\ \frac{e^{-\pi i s}\Gamma(1-s)}{2\pi i}\ 4\pi i\, e^{i\pi s}\sin\frac{\pi s}{2}\sum_{m=1}^{n}(2m\pi)^{s-1}$$

This formula is valid for all s. If $\sigma = \operatorname{Re}(s) < 0$, then

$$\int_{\gamma_n}\frac{z^{s-1}\,dz}{e^z-1} \longrightarrow 0, \qquad \sum_{m=1}^{n}(2m\pi)^{s-1} \longrightarrow (2\pi)^{s-1}\zeta(1-s)$$

as $n \longrightarrow \infty$.



The contour γ_n in the z-plane

Thus

IV $$\boxed{\zeta(s) = 2\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)(2\pi)^{s-1}\zeta(1-s)}$$

holds when $\sigma < 0$. By analytic extension the formula holds for all complex s. (!).

Using

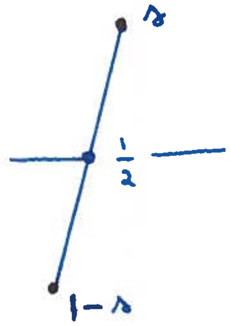
$$\Gamma\left(1-\tfrac{s}{2}\right)\Gamma\left(\tfrac{s}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi s}{2}\right)}$$ EULER'S REFLEXION FORMULA

$$\Gamma(1-s) = \frac{2^{-s}}{\sqrt{\pi}}\Gamma\left(\tfrac{1-s}{2}\right)\Gamma\left(1-\tfrac{s}{2}\right)$$ LEGENDRE'S DUPLICATION FORMULA

we finally have the functional equation

$$\boxed{\Gamma\left(\tfrac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\tfrac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}\zeta\left(\overset{1-s}{\cancel{\tfrac{1-s}{2}}}\right)}$$

when $s \in \mathbb{C}$ RIEMANN



for Riemann's zeta function.

Ex.: Take $s = 2$ in the functional equation.

$$\Gamma(1)\pi^{-1}\zeta(2) = \Gamma\left(-\tfrac{1}{2}\right)\pi^{1/2}\zeta(-1)$$

$$\zeta(s) = \sum \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \Gamma(1) = 0! = 1$$

$$\Gamma\left(-\tfrac{1}{2}\right) = -2\Gamma\left(\tfrac{1}{2}\right) = -2\sqrt{\pi}$$

$$\zeta(1) = -\frac{1}{12}$$ "$1+1+1+\cdots = -\frac{1}{12}$" Euler

Ex.: The Bernoulli numbers B_n appear in the expansion

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n z^n}{n!} = 1 - \frac{1}{2}z + \frac{1/6\, z^2}{2!} - \cdots$$

valid for $|z| < 2\pi$. When $s = -n$, a negative integer, the integrals along the real axis cancel in $\overline{\text{III}}$ because

$$e^{(s-1)[\ln x + 2i\pi]} = e^{(s-1)\ln x}, \qquad s-1 = -n-1$$

and

$$\zeta(-n) = \frac{e^{n\pi i}\,\Gamma(n+1)}{2\pi i} \oint_{|z|=\varepsilon} \frac{z^{-n-1}}{e^{z}-1}\,dz$$

$|z|=\varepsilon$

$$= \frac{(-1)^{n}\, n!}{2\pi i} \oint_{|z|=\varepsilon} \left(\sum_{k=0}^{\infty} \frac{B_k}{k!} z^{k-n-2} \right) dz$$

The residue is the coefficient of z^{-1}, which comes when $k-n-2=-1$

$$= (-1)^n n! \frac{B_{n+1}}{(n+1)!} = \frac{(-1)^n B_{n+1}}{n+1}$$

Now $B_3 = 0$, $B_5 = 0$, $B_7 = 0, \ldots$ so that we have the trivial zeros $\zeta(-2n) = 0$. The functional equation yields

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1} B_{2n}}{(2n)!\cdot 2}. \qquad (n=1,2,3,\ldots)$$

This was Riemann's first proof of the Functional Equation for $\zeta(s)$. He also gave a proof based on Poisson's formula

$$\sum_{k=-\infty}^{\infty} e^{-k^2\pi t} = \frac{1}{\sqrt{t}} \sum_{k=-\infty}^{\infty} e^{-\frac{k^2\pi}{t}}$$

k^{2} !

The location of the zeros of $\zeta(s)$ is important to estimate

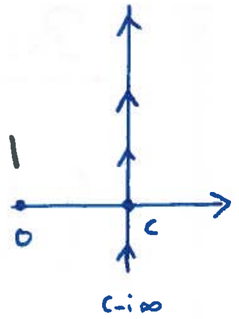
THE INTEGRAL

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

GOVERNS THE DISTRIBUTION OF PRIME NUMBERS

x^{c+1} is large!

$c > 1$



as $x \longrightarrow +\infty$. The formula requires that $\zeta(s) \neq 0$ when $\mathrm{Re}(s) \geq c$. The Prime Number Theorem is equivalent to $c = 1$, i.e., $\zeta(1+it) \neq 0$ when $-\infty < t < \infty$. The Riemann Hypothesis states that all the non-trivial zeros are located on the critical line $s = \frac{1}{2} + it$. So far, the Riemann Hypothesis has not been proved. A weaker one is the Lindelöf Hypothesis.

<u>Proof of II</u> Keep $\mathrm{Re}(s) > 1$.

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt = n^s \int_0^\infty e^{-nx} x^{s-1} dx$$

$$\Gamma(s)\zeta(s) = \Gamma(s)\sum n^{-s} = \sum_{n=1}^{\infty} \int_0^\infty e^{-nx} x^{s-1} dx$$

$$= \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-nx}\right) x^{s-1} dx = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}$$

The series was the geometric series! The series is absolutely convergent; hence the order of summation and integration can be switched. □

<u>Proof of III</u> Keep $\sigma = \mathrm{Re}(s) > 1$.

$$\int_{\gamma} \frac{z^{s-1}\,dz}{e^z - 1} = \underbrace{\int_{\infty}^{\varepsilon} \frac{x^{s-1}\,dx}{e^x - 1} + \int_{\varepsilon}^{\infty} \frac{(x e^{2\pi i})^{s-1}}{e^x - 1}\,dx}_{(e^{2\pi i s} - 1)\int_{\varepsilon}^{\infty} \frac{x^{s-1}\,dx}{e^x - 1}} + \oint_{|z|=\varepsilon} \frac{z^{s-1}\,dz}{e^z - 1}$$

The integral along the small circle vanishes as $\varepsilon \longrightarrow 0$, because $|e^z - 1| \geq \frac{1}{2}|z|$, when $|z|$ is small, and
(Taylor)

$$\left| \oint_{|z|=\varepsilon} \frac{z^{s-1}}{e^z - 1}\,dz \right| \leq \frac{\varepsilon^{\sigma-1} \cdot 2\pi\varepsilon}{\frac{1}{2}\varepsilon} = 4\pi\,\varepsilon^{\sigma-1} \longrightarrow 0.$$

It was essential that $\sigma > 1$. It follows that

$$\underbrace{\int_{\gamma} \frac{z^{s-1}\,dz}{e^z - 1}}_{\text{The value is independent of } \varepsilon,\ 0<\varepsilon<2\pi.\ \text{Cauchy.}} = 2i\,e^{\pi i s} \sin(\pi s) \int_{0}^{\infty} \frac{x^{s-1}\,dx}{e^x - 1}$$

$$\frac{\pi}{\sin(\pi s)} = \Gamma(s)\Gamma(1-s)$$

Thus III follows from I, after this calculation. The function

$$s \longmapsto \int_{\gamma} \frac{z^{s-1}\,dz}{e^z - 1}$$

is analytic for all complex s. (But <u>now</u> one cannot let $\varepsilon \longrightarrow 0$, if $\mathrm{Re}(s) < 1$!!!) $\square$