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Vår saksbehandler, innvalgstelefon

Vår dato – referanse

Deres dato – referanse

THE RIEMANN MAPPING THEOREM

Two arbitrary simply connected domains none of which is the whole plane can be conformally mapped onto each others. In symbols, there exists a bijective conformal map

$$f : \Omega_1 \longrightarrow \Omega_2 .$$

Also the inverse $f^{-1} : \Omega_2 \longrightarrow \Omega_1$ is conformal.

THEOREM Let Ω denote a simply connected domain, $\Omega \neq \mathbb{C}$. Fix a point $\zeta \in \Omega$. There exists a unique bijective conformal mapping f of Ω onto the unit disk $|w| < 1$ satisfying

$$f(\zeta) = 0, \quad f'(\zeta) > 0.$$

REMARKS 1°) Liouville's theorem about bounded entire functions excludes the case $\Omega = \mathbb{C}$. 2°) The annuli $r_1 < |z| < r_2$ and $R_1 < |w| < R_2$ can be conformally mapped onto each others only in the case $r_2/r_1 = R_2/R_1$. They are doubly connected domains. 3) In the general case, a rectangle cannot be mapped onto a square so that the four corners correspond. Nonetheless, three corners can be mapped on corners. (The case of two squares is an exception, of course.)

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4) The notation $f'(\zeta) > 0$ presupposes that $f'(\zeta)$ is a real number.

Nothing is said about the boundary correspondence. It is the interiors that are mapped. For example, it may happen that $\text{area}(\partial\Omega) > 0$. If the boundary of Ω is a Jordan curve, then it is possible to extend f to the boundary so that

$$f : \overline{\Omega} \longrightarrow \overline{\mathbb{D}} = \{w \mid |w| \leq 1\}$$

is a homeomorphism. This is the celebrated Carathéodory Extension Theorem (1913)

Riemann enunciated the proof in 1851 using Dirichlet's Principle for harmonic functions in his "proof". Dirichlet's Principle was proved by Hilbert 1900. The first rigorous proof of the Riemann Theorem is due to Koebe (19??). 1833?? (A "proof" can also be extracted from an earlier work of Osgood. 1900) Koebe avoided the Dirichlet Principle by posing an extremal problem over a "normal family" of functions.

Let us record some preliminary observations. If $|f'(z)| \leq L$ when $|z - z_0| < R$, then

$$|f(z_2) - f(z_1)| \leq L\,|z_2 - z_1|$$

when $|z_0 - z_j| < R$. This follows from

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(\zeta)\,d\zeta,$$

where we can integrate along a segment of a line, since the disk is convex.

If we have a bound on $|f(z)|$ in a domain, we can estimate $|f'(z)|$ in a smaller domain.

NOTATION $B(z_0, r) = \{ z \mid |z - z_0| < r \}$

LEMMA Suppose that $|f(z)| \le 1$ in Ω. If $\overline{B(z_0, 2r)} \subset \Omega$, then

$$|f'(z)| \le \frac{2}{r}, \quad \text{when } |z - z_0| \le r.$$

Proof: Use

$$f'(z) = \frac{1}{2\pi i} \oint_{|z_0 - \zeta| = 2r} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2},$$

take absolute values and observe that $\frac{1}{|\zeta - z|^2} \le \frac{1}{r^2}$.

Remark: With some extra consideration one can see that it is enough to assume that $f(z)$ is analytic in the interior $|z - z_0| < 2r$. (First, replace $2r$ by $2r - \varepsilon$ in the construction.)

THEOREM (Montel) Suppose that $\mathcal{F}$ is a family of analytic functions $f: \Omega \longrightarrow \mathbb{D}$. Then each sequence $f_1, f_2, \ldots, f_n, \ldots$ of functions belonging to $\mathcal{F}$ has a locally uniformly convergent subsequence.

NOTICE: $\sup_f |f|_\infty < \infty.$

REMARKS 1°) The uniform bound $|f(z)| \le 1$ for all $f \in \mathcal{F}$ is essential. (For the theorem it is sufficient that the family be <u>locally</u> uniformly bounded.) 2°) According to Weierstrass's theorem the limit functions are analytic. A limit function does not have to be member of the family.
3°) The theorem can be derived from the <u>Ascoli-Arzelà</u> theorem in Functional Analysis.

<u>Proof</u>: First, let $\zeta_1, \zeta_2, \zeta_3, \dots$ be a numbering of the rational points in Ω. Since the sequence

$$f_1(\zeta_1),\ f_2(\zeta_1),\ f_3(\zeta_1), \dots$$

is bounded, it has a convergent subsequence $f_{n_1}(\zeta_1)$, $f_{n_2}(\zeta_1)$, $f_{n_3}(\zeta_1), \dots$ Denote it by $f_{1n}(\zeta_1)$, $n = 1, 2, 3, \dots$ The sequence

$$f_{11}(\zeta_2),\ f_{12}(\zeta_2),\ f_{13}(\zeta_2), \dots$$

is bounded and again we can extract a convergent subsequence of numbers, say $f_{21}(\zeta_2)$, $f_{22}(\zeta_2)$, $f_{23}(\zeta_2), \dots$ The scheme is

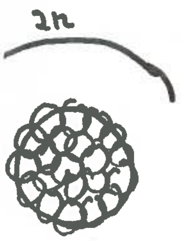
$$\left\{\begin{array}{llllll}
f_{11}, & f_{12}, & f_{13}, & f_{14}, \dots & \text{converges at} & \zeta_1 \\
f_{21}, & f_{22}, & f_{23}, & f_{24}, \dots & \text{— " —} & \zeta_2 \ (\text{and } \zeta_1) \\
f_{31}, & f_{32}, & f_{33}, & f_{34}, \dots & \text{— " —} & \zeta_3 \ (\text{and } \zeta_1, \zeta_2 \\
\vdots & \vdots & \vdots & \vdots & & \vdots
\end{array}\right.$$

Each sequence is a subsequence of all the previous ones. <u>The diagonal sequence</u>

$$g_n = f_{nn} \qquad (n = 1, 2, 3, \ldots)$$

<u>converges at every rational point</u> ζ_k !

Suppose that $B_{2r} = \{ z \mid |z - z_0| < 2r \}$ is in Ω. Let $\varepsilon > 0$ be given. The disk $\overline{B_r} = \{ z \mid |z - z_0| \le r \}$ can be covered by a finite number of disks Δ_k of radius $\delta < \frac{\varepsilon r}{2}$. Pick one rational point $w_k \in \Delta_k$, $k = 1, 2, \ldots, N$. Let $z \in B_r$ be arbitrary. Then $z \in \Delta_k$ for some k and



$$|g_n(z) - g_m(z)| \le |g_n(z) - g_n(w_k)| + |g_n(w_k) - g_m(w_k)| + |g_m(w_k) - g_m(z)|$$

LEMMA

$$\le \frac{2}{r} |z - w_k| + |g_n(w_k) - g_m(w_k)| + \frac{2}{r} |z - w_k|$$

$$\le \frac{4}{r} \cdot \frac{\varepsilon r}{2} \cdot 2 + \overbrace{|g_n(w_k) - g_m(w_k)|}^{\to 0} < 5\varepsilon$$

when $m, n > N_\varepsilon$. Thus we have a Cauchy sequence. Because only a finite number of w_k's is involved, N_ε is independent of the point z.

So far we have constructed a subsequence that converges uniformly in the disk $|z-z_0| \le r$, and we assumed that $|f_n(z)| \le 1$, when $|z-z_0| < 2r$. Suppose now that $K \subset \Omega$ is an arbitrary compact set. We can cover K with open disks $B(z, r_z)$ so that $B(z, 2r_z) \subset \Omega$. By the compactness of K, a finite number of disks will do:

$$K \subset \bigcup_{j=1}^{M} B(z_j, r_{z_j}), \quad B(z_j, 2r_{z_j}) \subset \Omega.$$

We can perform the previous construction to extract a uniformly convergent subsequence in the first disk. Then, starting with this subsequence, we extract a subsequence again, this time converging uniformly in the second disk. Thus we have uniform convergence in two disks. Proceeding inductively, we obtain a subsequence converging uniformly in the union of the disks, and, a fortiori, in K. To conclude the proof, let

$$K_n = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \ge \frac{1}{n}, \ |x| \le n\}$$

so that $K_1 \subset K_2 \subset K_3 \subset \cdots$ There is a subsequence converging uniformly in the compact set K_1. Extract a subsequence converging uniformly in K_2. Etc. Again, use the procedure of diagonalization to obtain a subsequence converging uniformly in each fixed K_n. ∎

Proof of Riemann's Mapping Theorem:

The uniqueness is reduced to the fact that the only conformal mapping of the unit disk onto itself, keeping the origin fixed, is a rotation (Schwarz's lemma). The condition $f'(\zeta) > 0$ forces it to be the identity. (Look at $f_2 \circ (f_1^{-1})$ to see that $f_1 = f_2$.)

For the proof of existence we consider the family $\mathcal{F}$ consisting of all functions $g(z)$ such that

(i) $g(z)$ is analytic and univalent in Ω,

(ii) $|g(z))| \leq 1$ in Ω,

(iii) $g(\zeta) = 0$, $g'(\zeta) > 0$ (ζ is given).

The proof consists of three parts:

- The family $\mathcal{F}$ is not empty
- There is a $f \in \mathcal{F}$ such that

$$f'(\zeta) = \sup_{g \in \mathcal{F}} g'(\zeta) \overset{\text{DEF}}{=} B$$

- This f is the desired mapping

$\mathcal{F} \neq \phi$ (This is easy, if the complement of Ω contains a disk.) There is a point $a \neq \infty$ in the complement of Ω. Thus $z - a \neq 0$ when $z \in \Omega$. Since Ω is simply connected we can define a single-valued branch

$$h(z) = \sqrt{z-a}$$

to "open up" the complement. We have

$$h(z_1) \neq \pm h(z_2), \text{ when } z_1 \neq z_2, \ z_1, z_2 \in \Omega$$
$$h(z) \neq - h(z)$$

The image $h(\Omega)$ contains a disk $|w - h(\zeta)| < \rho$ and therefore $|h(z) + h(\zeta)| \geq \rho$ when $z \in \Omega$ and in particular $2|h(\zeta)| \geq \rho$. It can now be verified that

$$g(z) = \frac{\rho}{4} \frac{|h'(\zeta)|}{|h(\zeta)|^2} \frac{h(\zeta)}{h'(\zeta)} \frac{h(z) - h(\zeta)}{h(z) + h(\zeta)}$$

belongs to the family $\mathcal{F}$. Indeed, it is univalent, $g(\zeta) = 0$, and

$$g'(\zeta) = \frac{\rho}{8} \left| \frac{h'(\zeta)}{h(\zeta)^2} \right| > 0.$$

The estimate $|g(z)| \leq 1$ comes from a calculation!

Let us turn to the extremal problem. There is a sequence of functions $f_n \in \mathcal{F}$ such that

$$\lim_{n \to \infty} f_n'(\zeta) = B = \sup_{g \in \mathcal{F}} g'(\zeta).$$

By Montel's theorem there is a locally uniformly convergent subsequence $f_{n_1}, f_{n_2}, \ldots$ By Weierstrass's theorem the limit function f is analytic. Moreover, $f_n' \longrightarrow f'$ locally uniformly in Ω. Thus

$$f'(\zeta) = B \,, \quad |f(z)| \le 1$$

(This also shows that $B < \infty$.) By Hurwitz's Theorem $f(z)$ is either univalent or a constant. Since $f'(\zeta) = B \neq 0$ the case of a constant is ruled out. We have obtained an extremal function $f \in \mathcal{F}$.

It remains to show that $f(z)$ takes every value w with $|w| < 1$. Suppose that $f(z)$ omits the value ω, where $|\omega| < 1$. Hence we can define a single-valued branch of

$$F(z) = \sqrt{\frac{f(z) - \omega}{1 - \overline{\omega} f(z)}}$$

It is clear that $F(z)$ is univalent and that $|F(z)| \le 1$. The normalized function

$$G(z) = \frac{|F'(\zeta)|}{F'(\zeta)} \frac{F(z) - F(\zeta)}{1 - \overline{F(\zeta)}\, F(z)}$$

belongs to $\mathcal{F}$! A calculation yields

NOTICE THAT $\omega \neq 0$.

$$G'(\zeta) = \frac{1 + |\omega|}{2\sqrt{|\omega|}} B > B ,$$

a contradiction to the defining property of B. Hence ω is not an omitted value. Thus $f(z)$ takes all values w with $|w| < 1$.

We had $|f(z)| \le 1$, but by the Maximum Principle $|f(z)| < 1$, when $z \in \Omega$.

THE END

__Lemma__ If $f(z) \neq 0$ in the simply connected domain Ω, a single-valued branch of $\log f(z)$ can be defined in Ω.

__Proof:__ The desired function is

$$\log(f(z)) = \int_{z_0}^{z} \frac{f'(\zeta)\,d\zeta}{f(\zeta)} + \log(f(z_0))$$

because the integral is path-independent. Because

$$\frac{d}{dz}\left(f(z)\, e^{-\int^{z} \cdots d\zeta}\right) = \cdots = 0,$$

one can easily verify that this __is__ the logarithm, i.e. $\exp(\log f(z)) = f(z)$, as it should. □

$$\log f(z) = \log|f(z)| + i \underbrace{\arg f(z)}$$

THIS IS ALSO DEFINED !

$$\sqrt{f(z)} = e^{\frac{1}{2}\log(f(z))}$$

$$= |f(z)|^{1/2} e^{i\frac{\arg f(z)}{2}}$$

__Remark:__ In the proof of the Riemann Mapping Thm the fact that Ω was __simply__ connected, was used only for the extraction of square roots!

SIMPLY CONNECTED