COMPLEX ANALYSIS 24. V. 2018

(1) It is necessary that $\Delta u = 0$, i.e.,

$$\Delta(ax^3 + 3bx^2y + 3xy^2 + 2y^3)$$
$$= 6ax + 6by + 6x + 12y = 0$$

Hence $\underline{a = -1}$, $\underline{b = -2}$.

$$f = (-x^3 - 6x^2y + 3xy^2 + 2y^3) + i\,v(x,y)$$

The Cauchy - Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

yield v.

<u>Answer</u>: $f(z) = (2i-1)\,z^3 + \text{Const.}$

(2) $$f(z) = \frac{1 + iz^2}{1 - iz^2}$$ will do.

(5) $|f(z)| \leq 2018\,|\sin(z)|$ Now $f(n\pi) = 0$

so that

$$h(z) = \frac{f(z)}{\sin(z)} \quad \text{(divide out zeros)}$$

is analytic and bounded $(|h(z)| \leq 2018)$. By Liouville's Theorem $h(z)$ is constant. Thus

$$f(z) = C \sin(z)$$

where $|C| \leq 2018$.

(4) By Cauchy's formula

$$a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{(z-0)^{n+1}}\, dz$$

$$|a_n| \le \frac{1}{2\pi} \int_{|z|=r} \frac{\frac{1}{1-|z|}}{|z|^{n+1}}\,|dz| \le \frac{r}{r^{n+1}(1-r)}$$

where $0 < r < 1$. The optimal r is

$r = \frac{n}{n+1} = 1 - \frac{1}{n+1}$. Then

$$|a_n| \le \frac{1}{(n+1)^{-1}\left(\frac{n}{n+1}\right)^n} = (n+1)\left(1+\frac{1}{n}\right)^n$$

(6) By Schwarz lemma $|g(z)| \le |z|$ and so $|g(z^k)| \le |z^k| = |z|^k$. Now

$$\sum_{k=1}^{\infty} |g(z^k)| \le \sum_{k=1}^{\infty} |z|^k = \frac{|z|}{1-|z|} < \infty.$$

Hence the series converges even absolutely.

(7) The product converges when the series

$$\sum_{k=1}^{\infty} \underbrace{\log\left\{\left(1+\frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} + h(z)\right)\right\}}_{\underbrace{k\log\left(1+\frac{z}{k}\right)}_{z - \frac{z^2}{2k} + O\left(\frac{1}{k^2}\right)} + \frac{z^2}{2k} + h(z)}$$

converges.

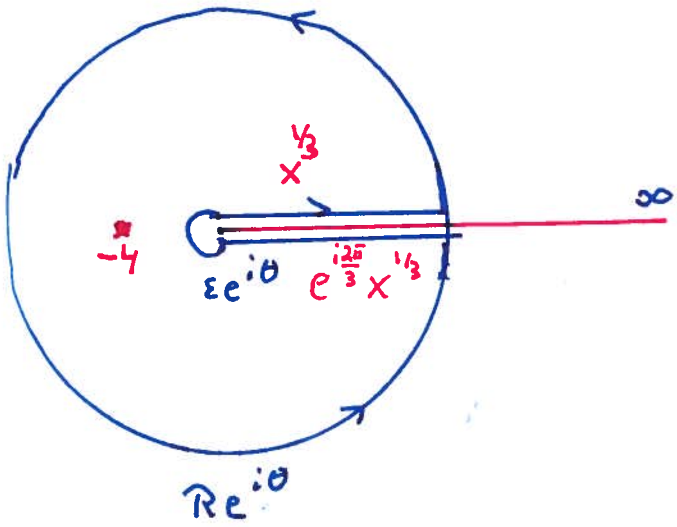
$$\boxed{\text{Thus, take } h(z) = -z \text{ for convergence.}}$$

(3) $z^{1/3} = (re^{i\theta})^{1/3}$

$= r^{1/3} e^{i\theta/3}$

when $0 < \theta < 2\pi$

Key hole contour



The integrals along the circles

$|z| = R \longrightarrow \infty$

$|z| = \varepsilon \longrightarrow 0+$

approach zero. There is a double pole at $z = -4$.

By the Residue Theorem

$$\int_0^\infty \frac{x^{1/3}\,dx}{(4+x)^2} - e^{\frac{2\pi i}{3}} \int_0^\infty \frac{x^{1/3}\,dx}{(4+x)^2} = 2\pi i \operatorname*{Res}_{z=-4}\left\{\frac{z^{1/3}}{(4+z)^2}\right\}$$

$$\operatorname{Res}\left\{\frac{z^{1/3}}{(4+z)^2}\right\} = \frac{d}{dz} z^{1/3}\Big|_{z=-4} = \frac{1}{3}\left(\frac{z^{1/3}}{z}\right)_{z=-4}$$

$$= -\frac{1}{12} 4^{1/3} e^{i\pi/3}$$

$$\int_0^\infty \frac{x^{1/3}\,dx}{(4+x)^2} = \frac{2^{2/3}\,2\pi i}{12} \frac{e^{i\pi/3}}{e^{2\pi i/3} - 1} = \frac{2^{2/3}}{12} \frac{2\pi i}{e^{i\pi/3} - e^{-i\pi/3}}$$

$$= \frac{\pi\, 2^{2/3}}{12 \sin(\frac{\pi}{3})} = \frac{\pi}{2^{1/3}\, 3\sqrt{3}}$$