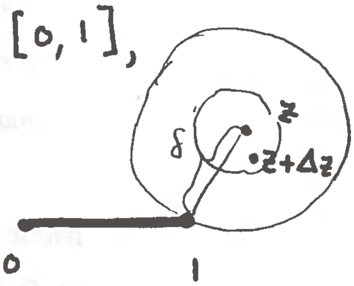
EXAMPLES: §II.1 no 14, §II.2 no 6, §II.1 no 5

Suppose, for simplicity, that $h(t)$ is real-valued and continuous, when $0 \le t \le 1$. ① Then

$$H(z) = \int_0^1 \frac{h(t)}{t-z}\,dt, \quad z \in \mathbb{C} \setminus [0,1],$$



is continuous. Proof: Fix $z$ and denote

$$\delta = \min_{0 \le t \le 1} |t-z| > 0.$$

Let $|\Delta z| < \frac{\delta}{2}$. Then $|t-z-\Delta z| \ge |t-z| - |\Delta z| \ge \delta - \frac{\delta}{2} = \frac{\delta}{2}$. Now

$$|H(z+\Delta z) - H(z)| = \left| \Delta z \int_0^1 \frac{h(t)\,dt}{(t-z)(t-z-\Delta z)} \right|$$

$$\le |\Delta z| \int_0^1 \frac{|h(t)|\,dt}{|t-z|\,|t-z-\Delta z|} \le \frac{2|\Delta z|}{\delta^2} \max_{0 \le t \le 1} |h(t)|$$

$\longrightarrow 0$ as $\Delta z \to 0$. This is the desired continuity

$$\lim_{\Delta z \to 0} H(z+\Delta z) = H(z).$$

② Claim: $H'(z) = \int_0^1 \frac{h(t)\,dt}{(t-z)^2}$

Proof: A similar calculation yields for $\Delta z \ne 0$

$$\left| \frac{H(z+\Delta z) - H(z)}{\Delta z} - \int_0^1 \frac{h(t)\,dt}{(t-z)^2} \right| = \left| \Delta z \int_0^1 \frac{h(t)\,dt}{(t-z)^2 (t-z-\Delta z)} \right|$$

$$\le \frac{2 \cdot |\Delta z|}{\delta^3} \max_{0 \le t \le 1} |h(t)| \longrightarrow 0 \text{ as } \Delta z \to 0.$$

③ The derivative $H'(z)$ is continuous, $z \notin [0,1]$.

The proof requires a justification of

$$\lim_{\Delta z \to 0} \int_0^1 \frac{h(t)}{(t-z-\Delta z)^2} = \int_0^1 \frac{h(t)}{(t-z)^2},$$

which can be based on the formula

$$\frac{1}{(t-z-\Delta z)^2} - \frac{1}{(t-z)^2} = \frac{\Delta z \cdot (2t-2z-\Delta z)}{[(t-z)(t-z-\Delta z)]^2}.$$

As before

$$|H'(z+\Delta z) - H'(z)| \leq \frac{4|\Delta z|(2+2|z|+|\Delta z|)}{\delta^4} \max|h|.$$

$$\longrightarrow 0 \text{ as } \Delta z \longrightarrow 0.$$

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§ $\frac{\text{II}}{5}$.1 Another proof of the existence of the limit

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n\right) = 0{,}5772157\ldots$$

(Euler's gamma) is based on the series $\sum u_n$, where

$$0 < u_n = \int_0^1 \frac{t\,dt}{n(n+t)} < \int_0^1 \frac{dt}{n^2} = \frac{1}{n^2}$$

The series $\sum u_n$ converges, because the majorant $\sum 1/n^2$ does. By integration

$$u_n = \int_0^1 \frac{dt}{n} - \int_0^1 \frac{dt}{n+t} = \frac{1}{n} - \log\left(\frac{n+1}{n}\right).$$

$- [\log \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n+1}{n}$

Now

$$\sum_{j=1}^{\infty} u_j = \lim_{n\to\infty}\sum_{j=1}^{n} u_j = \lim_{n\to\infty}\left[\sum_{j=1}^{n} u_j + \log\frac{n+1}{n}\right] = \lim_{n\to\infty}\left[1+\frac{1}{2}+\dots+\frac{1}{n}-\log n\right]$$

Thus $\gamma = \sum_{j=1}^{\infty} u_j$, as desired.