① Suppose that $f = u + iv$ is analytic in Ω. Find the Laplacian:

$$\Delta(uv) = ?$$

② Assume that $f(z) \neq 0$ in the <u>simply</u> <u>connected</u> domain Ω. Consider the function

$$h(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta, \qquad (z_0 \in \Omega)$$

which is independent of the path γ joining z_0 and z in Ω. Calculate

$(\ln = \log)$

$$\frac{d}{dz}\left(e^{-h(z)} f(z)\right)$$

and conclude that $h(z) = \ln(f(z))$.

(Since

$$\ln(f(z)) = \ln(|f(z)|) + i \arg(f(z)),$$

also a <u>single</u> <u>valued</u> branch of $\arg(f(z))$ can be defined in Ω.)

<u>Remark</u> $e^w = z \neq 0 \Longleftrightarrow w = \ln(z), \quad z \neq 0$

$\Longleftrightarrow w = \ln(|z|) + i(\theta + 2n\pi), \quad z \neq 0.$

③ Map the union of the disks $|z| < 1$ and $|z-1| < 1$ conformally on the unit disk.

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(4) Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial. It has n roots (zeros), counted according to multiplicity, by the Fundamental Theorem of Algebra. Show that

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{P'(z)}{P(z)}\, dz = \text{the number of roots inside the simple loop } \gamma.$$

(This is a special case of the Principle of Argument.)