① The series

$$\sum_{n=0}^{\infty} \frac{\sin(nz)}{2^n} = \frac{2\sin(z)}{5-4\cos(z)}$$

converges if and only if $z$ belongs to the strip $|y| < \ln(2)$, $z = x+iy$. Hint: With Euler's formula

$$\sin(nz) = \frac{e^{inz} - e^{-inz}}{2i}$$

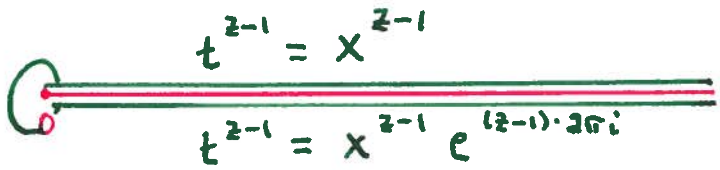
the problem is reduced to geometric series.

③ $t^{z-1} = e^{(z-1)\log(t)}$, $\log(t) = \log|t| + i\arg(t)$

Caution $t$ is complex, $z \neq x+iy$ here!!!

$0 < \arg(t) < 2\pi$

Re(z) > 0




$$I(z) = +\int_{\infty}^{\varepsilon} \underset{\text{upper strand}}{x^{z-1} e^{-x}\,dx} + e^{2\pi i z} \int_{\varepsilon}^{\infty} \underset{\text{lower strand}}{x^{z-1} e^{-x}\,dx}$$

$$+ \underbrace{\int_0^{2\pi} (\varepsilon e^{i\theta})^{z-1} e^{-\varepsilon e^{i\theta}} i\varepsilon e^{i\theta}\,d\theta}_{\text{abs value } \leq \varepsilon^{Re(z)} \cdot 2\pi \longrightarrow 0 \text{ as } \varepsilon \to 0.}$$

The circle $|t| = \varepsilon$

$$I(z) = (e^{2\pi i z} - 1) \underbrace{\int_0^{\infty} t^{z-1} e^{-t}\,dt}_{= \Gamma(z)}$$

NTNU
Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag
7491 Trondheim

(4) <u>Misprint</u>: write $|z|$ in place of $x$.

$$\left|e^{z}-1-z-\frac{z^2}{2}\right| = \left|\sum_{n=3}^{\infty}\frac{z^n}{n!}\right| \le \sum_{n=3}^{\infty}\frac{|z|^n}{n!}$$

$$= e^{|z|}-1-|z|-\frac{|z|^2}{2}.$$

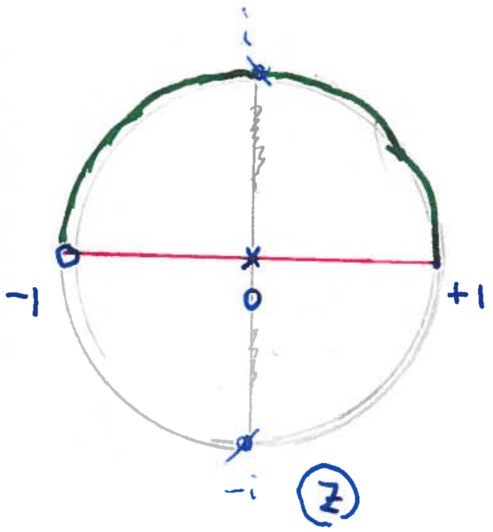
(5) circle → ellips:

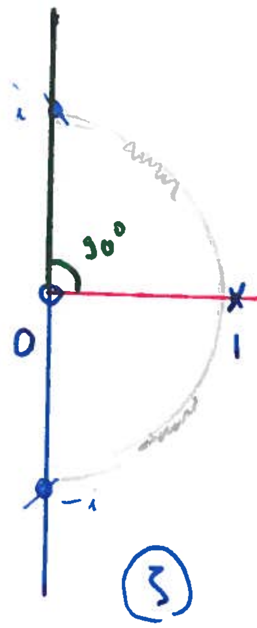

$w = \frac{1}{2}\left(z+\frac{1}{z}\right)$ (Joukowski transform)

---

(9) It is convenient to write

$$w = \frac{1}{4}\left\{\left(\frac{1+z}{1-z}\right)^2-1\right\}, \quad |z|<1$$

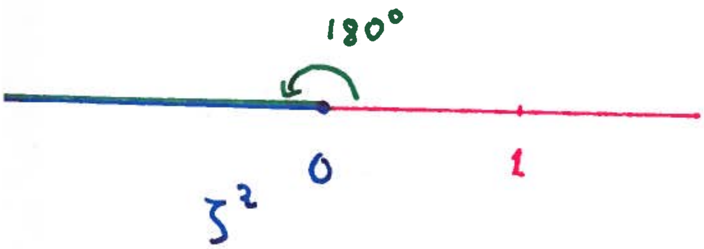



MÖBIUS

$$\zeta = \frac{1+z}{1-z}$$

$|z|<1$ is mapped onto the half-plane $\mathrm{Re}(\zeta)>0$





<u>Answer</u>: $\mathbb{C}\setminus(-\infty,-\frac{1}{4}]$

NTNU
Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag
7491 Trondheim

NTNU
Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag
7491 Trondheim

(6) By default, $f(x) = 0$ when $x < 0$. (Use the <u>right</u> half-circle to see this.) Write $\zeta$ in place of $s$.

$$\operatorname*{Res}_{\zeta=0}\left\{\frac{e^{\zeta x}}{\zeta(\zeta+1)}\right\} = 1, \quad \operatorname*{Res}_{\zeta=-1}\left\{\frac{e^{\zeta x}}{\zeta(\zeta+1)}\right\} = -e^{-x}$$

On the left half-circle

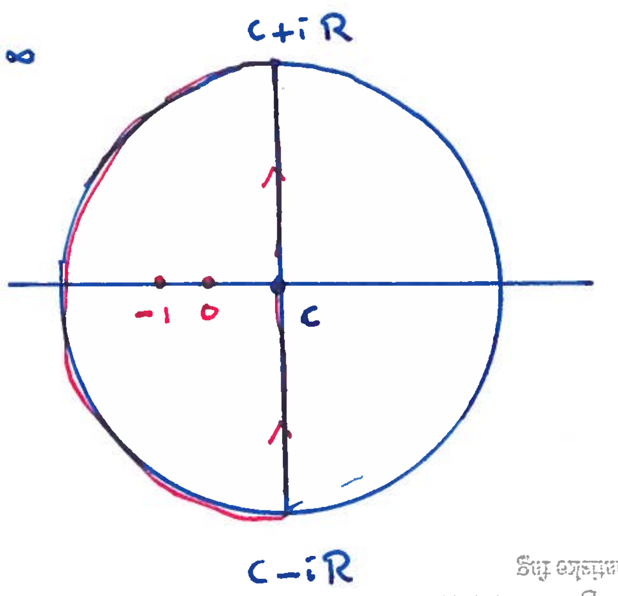
$$|e^{\zeta x}| \le e^{cx} \quad (x \ge 0) \quad \left|\frac{1}{\zeta(\zeta+1)}\right| \le \frac{1}{(R-c)(R-c-1)}$$

and

$$\left|\frac{1}{2\pi i}\int_{\substack{|\zeta-c|=R\\ \operatorname{Re}(\zeta)\le c}} F(\zeta)e^{\zeta x}\,d\zeta\right| \le \frac{\cancel{2}\pi R}{2\pi}\,\frac{e^{cx}}{(R-c)(R-c-1)} \xrightarrow{R\to\infty} 0.$$

Thus

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{sx}}{s(s+1)}\,ds = 1 - e^{-x}$$




NTNU
Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag
7491 Trondheim