BOUNDARY VALUES IN POISSON'S FORMULA.

Let $g = g(\theta)$ be continuous (and periodic).

$$u(r,\theta) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{g(\phi)}{R^2 - 2rR\cos(\theta - \phi) + r^2}\, d\phi$$

POISSON's formula

<u>Claim:</u> $\displaystyle \lim_{\substack{(r,\theta)\to(R,\theta_0)\\ r<R}} u(r,\theta) = g(\theta_0)$

<u>Remark:</u> $\Delta u = 0$

<u>Proof</u> (without Complex Analysis): By direct computation

$$\frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{d\phi}{R^2 - 2rR\cos(\theta - \phi) + r^2} = 1$$

when $r < R$. Let $\varepsilon > 0$. By continuity we fix $\delta_\varepsilon > 0$ so small that

$$|g(\phi) - g(\theta_0)| < \varepsilon \quad \text{when} \quad |\phi - \theta_0| < \delta_\varepsilon .$$

Now

$$u(r,\theta) - g(\theta_0) = \frac{R^2 - r^2}{2\pi} \int \frac{g(\phi) - 1\, g(\theta_0)}{R^2 - 2rR\cos(\theta - \phi) + r^2}\, d\phi$$

$$= \frac{R^2 - r^2}{2\pi} \int_{|\phi - \theta_0| < \delta_\varepsilon} \cdots\, d\phi \; + \; \frac{R^2 - r^2}{2\pi} \int_{|\phi - \theta_0| \geq \delta_\varepsilon} \cdots\, d\phi$$

Taking abs. values we get, when also

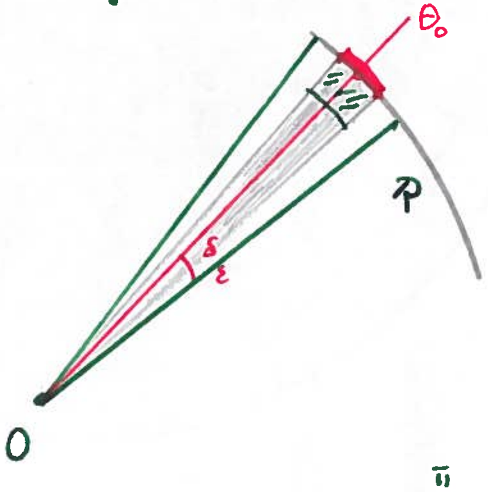
θ is close to θ_0, say $|\theta - \theta_0| < \frac{1}{2}\delta_\varepsilon$,
so that $|\theta - \phi| \geq \frac{1}{2}\delta_\varepsilon$ when $|\phi - \theta_0| \geq \delta_\varepsilon$,

$$|u(r,\theta) - g(\theta_0)| \leq \varepsilon \frac{R^2 - r^2}{2\pi} \int_{|\phi-\theta_0|<\delta_\varepsilon} \frac{d\phi}{R^2 - 2rR\cos(\theta-\phi) + r^2}$$

$$+ 2\,\mathrm{Max}|g(\theta)| \frac{R^2-r^2}{2\pi} \int_{|\phi-\theta_0|\geq\delta_\varepsilon} \frac{d\phi}{\underbrace{R^2 - 2rR\cos(\theta-\phi) + r^2}_{\geq R^2 - 2rR\cos\left(\frac{\delta_\varepsilon}{2}\right) + r^2}}$$

$$\leq \varepsilon \cdot 1 + \underbrace{2\,\mathrm{Max}|g(\theta)| \left[\frac{R^2 - r^2}{R^2 - 2rR\cos\left(\frac{\delta_\varepsilon}{2}\right) + r^2}\right]}_{\longrightarrow 0 \text{ as } r \longrightarrow R^-}$$

$< \varepsilon + \varepsilon$ when $|\theta - \theta_0| < \frac{\delta_\varepsilon}{2}$ and
$R - \gamma_\varepsilon < r < R$ for some $\gamma_\varepsilon \approx 0$.



(6) ? = 0. Use

$$I = \int_0^\pi \log(\sin x)\,dx = -\pi \log(2)$$

$$I = \int_0^\pi \log\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)dx = \pi\ln(2) + \underbrace{\int_0^\pi \ln\left(\sin\frac{x}{2}\right)dx}_{2\int_0^{\pi/2}\ln(\sin x)\,dx} + \underbrace{\int_0^\pi \ln\left(\cos\frac{x}{2}\right)dx}_{=\int_0^\pi \ln\left(\sin\frac{x}{2}\right)dx}$$

ETC.