COMPLEX ANALYSIS /7

2018

① Where do all the zeros go? Show that the polynomial

$$1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

has no zeros in the disk $|z| < R$, when n is large enough.

② Prove <u>Hurwitz's</u> <u>Theorem</u> for analytic functions. Let $f_k \longrightarrow f$ locally uniformly in Ω. Suppose that $f(z_0) = 0$ but $f(z) \not\equiv 0$. If the order of the zero z_0 is n, then each sufficiently small disk $|z - z_0| < \rho$ contains exactly n zeros of $f_k(z)$, provided that k is large enough.

③ Show that

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}$$

converges locally uniformly in the disk $|z| < 1$. Find $f'''(0)$ and $f^{(17)}(0)$.

④ Let $f(z)$ be analytic in the whole complex plane (= an ENTIRE function). Assume that $\mathrm{Re}(f(z)) \leq a < \infty$ when $z \in \mathbb{C}$. Prove that $f(z) \equiv \text{Constant}$.

⑤ Assume that $f(z)$ is analytic and that

$$f'(z_0) = a_1 \neq 0.$$

Show that $f(z)$ is univalent (injective) in a small neighbourhood $|z-z_0| < \delta$.

⑥ Find the inverse Fourier transform of $(1+i\omega)^{-2}$. That is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega x}}{(1+i\omega)^2}\, d\omega = \cdots\cdots = \begin{cases} 0, & x < 0 \\ xe^{-x}, & x > 0 \end{cases}$$