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COMPLEX ANALYSIS / 10

We know that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \qquad \gamma = 0,5772\ldots$$

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

① Prove

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad \text{REFLEXION FORMULA}$$

② $$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \qquad \text{DUPLICATION FORMUL}$$

LEGENDRE

③ Verify that $\boxed{\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty}$, where the p_k are the primes $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$. Hint:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\prod_{k=1}^{\infty} (1 - p_k^{-s})}, \quad s > 1 \qquad \text{(EULER)}$$

④ Poisson's Formula

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} g(\phi)\, d\phi$$

yields a harmonic function in the disk $r < R$.

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Assume that $g(\phi)$ is continuous and periodic. Prove that

$$\lim_{r \to R^-} u(r, \theta) = g(\theta).$$

<u>Hint:</u> If $g(\theta) \equiv 1$, also $u(r,\theta) \equiv 1$ by direct calculation.

⑤ If $f(0) \neq 0$ we have Jensen's formula

$$\log|f(0)| + \sum_{k=1}^{N} \log\left(\frac{R}{|z_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

with the zeros $z_1, \ldots, z_N$; $0 < |z_k| < R$. Write a formula for the case

$$f(z) = z^m c_m + \cdots \quad (m \geq 1 \text{ an integer}).$$

⑥ $$\int_0^{2\pi} \log|1 - e^{i\theta}|\,d\theta = ?$$

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