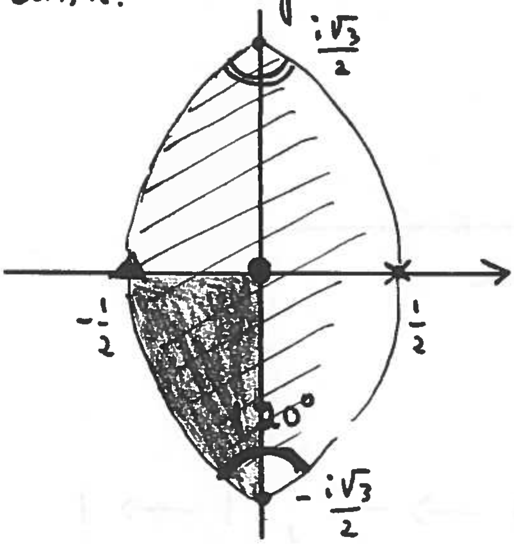
AHLFORS, Exercise I, p. 36 (Chapter 3, §4.2) P. Ly. 2000

The union of the disks $|z|<1$ and $|z-1|<1$ has to be mapped on the unit disk. 1) To preserve some symmetry, first perform the translation



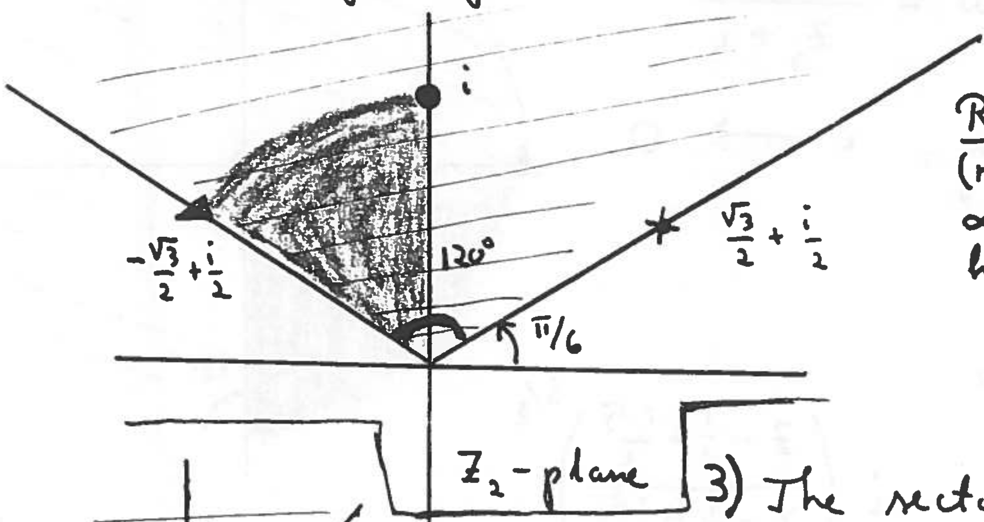
$$z_1 = z - \frac{1}{2}$$

Notice, the angle is $120°$ (not $60°$!).

2) $$z_2 = -i\,\frac{z_1 + \frac{i\sqrt{3}}{2}}{z_1 - \frac{i\sqrt{3}}{2}}$$

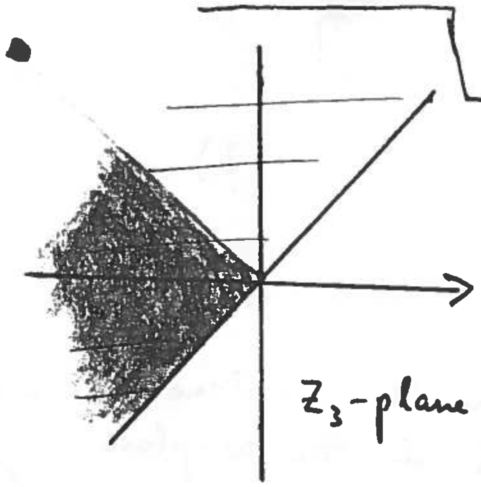
$$-\frac{i\sqrt{3}}{2} \to 0, \quad 0 \to i, \quad \frac{i\sqrt{3}}{2} \to \infty$$

Hence the imaginary axis is mapped on itself.



<u>Remark</u>: Half-lines (rays) through 0 and ∞ will be mapped on half-lines, when $z_2 \to z_3$

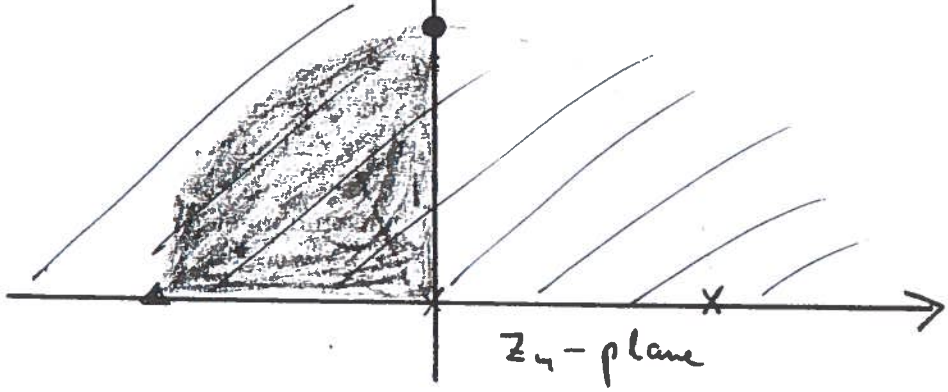
3) The sector is now mapped to a half-plane.



$$z_3 = z_2^{3/2} = +r_2^{3/2} e^{i\frac{3\theta_2}{2}}$$

Then $120°$ becomes $180°$!

4) Multiplying by $e^{-i\pi/4} = \frac{1-i}{\sqrt{2}}$ we get the more convenient situation



$$z_4 = e^{-i\pi/4} z_3$$

5) Let us try $-1 \to -1$, $0 \to -i$, $1 \to 1$

$$\frac{w+1}{w-1} = K \frac{z_4+1}{z_4-1}$$

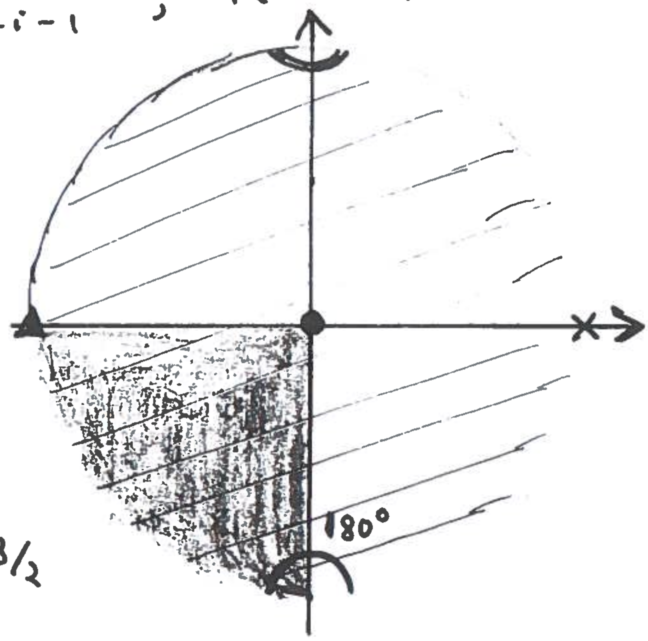
. For $z_4 = 0$ we get

$$-K = \frac{-i+1}{-i-1}, \quad K = -i.$$

Thus

$$w = \frac{i z_4 + 1}{z_4 + i}$$

Accidentally, $i \longrightarrow 0$



In toto,

$$w = \frac{1 - i\left(\dfrac{z - \frac{1}{2} + \frac{i\sqrt{3}}{2}}{z - \frac{1}{2} - \frac{i\sqrt{3}}{2}}\right)^{3/2}}{i - \left(\dfrac{z - \frac{1}{2} + \frac{i\sqrt{3}}{2}}{z - \frac{1}{2} - \frac{i\sqrt{3}}{2}}\right)^{3/2}} \qquad (?)$$

Circles through the points $\pm \frac{i\sqrt{3}}{2}$ in the z-plane $(|z| \leq 1)$ become circles through the points $\pm i$ in the w-plane.