Peter Lindqvist 2018

# BARRIERS

Poincaré

A barrier is an auxiliary function the existence of which guarantees that the correct continuous boundary values $f : \partial\Omega \longrightarrow \mathbb{R}$ are attained in the Dirichlet boundary value problem

$$\begin{cases} \Delta h = 0 \quad \text{in } \Omega \\ \lim\limits_{x \to \xi} h(x) = f(\xi) \quad \text{when } \xi \in \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$. Let

$$m = \min_{\xi \in \partial\Omega}\{f(\xi)\}, \quad M = \max_{\xi \in \partial\Omega}\{f(\xi)\}.$$

Recall the <u>Perron solutions</u>

$$\overline{H}(x) = \inf_{v \in \mathcal{U}} v(x), \quad \underline{H}(x) = \sup_{u \in \mathcal{L}}\{u(x)\},$$

$$m \leq \underline{H}(x) \leq \overline{H}(x) \leq M$$

Here the <u>upper class</u> $\mathcal{U}$ consists of all superharmonic functions $v : \Omega \longrightarrow (-\infty, \infty]$ such that

$$\liminf_{x \to \xi}(v(x)) \geq f(\xi) \quad \text{when } \xi \in \partial\Omega.$$

DEF The function $\omega : \Omega \longrightarrow (0, \infty]$ is a <u>barrier</u> at the point $\xi_0 \in \partial\Omega$, if

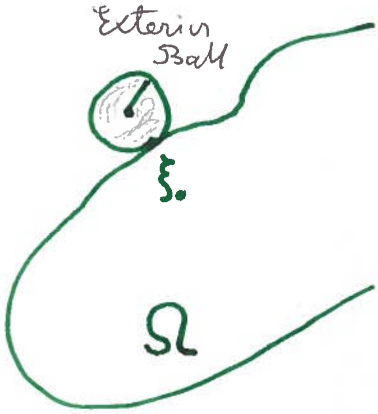
1) $\omega > 0$ and superharmonic in $\Omega$,

2) $\liminf\limits_{x \to \xi} \omega(x) > 0$ when $\xi \in \partial\Omega$, $\xi \neq \xi_0$,

3) $\lim\limits_{x \to \xi_0} \omega(x) = 0.$

In particular, if $\omega \in C(\overline{\Omega})$ the requirement is that $\omega(\xi_0) = 0$, $\omega(\xi) > 0$ when $\xi \neq \xi_0$.

<u>Ex.</u> [Exterior Sphere Condition]. Let $\xi_0 \in \partial\Omega$ and assume that there exists a ball (disk) $B(y_0, r) \subset \mathbb{R}^n \setminus \Omega$ tangent at $\xi_0$: $|y_0 - \xi_0| = r$. Then




$$\omega(x) = \frac{1}{r^{n-2}} - \frac{1}{|x - y_0|^{n-2}}$$

will do as a barrier at $\xi_0$, if $n \geq 3$. (The case $n = 2$ ?)

THEOREM If there exists a barrier at the point $\xi_0 \in \partial\Omega$, then

$$\lim_{x \to \xi_0} \underline{H}(x) = f(\xi_0) = \lim_{x \to \xi_0} \overline{H}(x).$$

<u>Proof</u>. Let $\varepsilon > 0$. There is $\delta_\varepsilon > 0$ ($f$ continuous) such that

$$|f(\xi) - f(\xi_0)| < \varepsilon \quad \text{when} \quad |\xi - \xi_0| < \delta_\varepsilon.$$

Now $\omega_\delta \equiv \inf\limits_{\substack{|x-\xi_0| \geq \delta_\varepsilon \\ x \in \Omega}} \omega(x) > 0$ so that we can determine $\lambda >> 1$ for which

$$\lambda\,\omega(x) \geq 2M \quad \text{when} \quad |x - \xi_0| \geq \delta_\varepsilon.$$

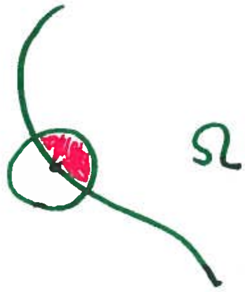
We have

$$\underbrace{\overset{\text{subharmonic}}{f(\xi_0) - \varepsilon - \lambda\omega(x)}}_{\text{Belongs to } \mathfrak{L}} \le \underline{H}(x) \le \overline{H}(x)$$

$$\le \underbrace{\overset{\text{superharmonic}}{f(\xi_0) + \varepsilon + \lambda\omega(x)}}_{\text{Belongs to } \mathfrak{U}}$$

For example, it is evident that $v(x) = f(\xi_0) + \varepsilon + \lambda\omega(x)$ belongs to the upper class:

$$\underline{|x - \xi_0| < \delta} \Rightarrow v(x) \ge (f(\xi) - \varepsilon) + \varepsilon + 0$$
$$= f(\xi)$$




$$\underline{|x - \xi_0| \ge \delta} \Rightarrow v(x) \ge$$

$$f(\xi_0) + \varepsilon + 2M \ge$$
$$\ge \varepsilon + M \ge f(\xi)$$

Since $\omega(x) \xrightarrow[x \to \xi_0]{} 0$ by assumption, we obtain

$$f(\xi_0) - 2\varepsilon \le \underline{H}(x) \le \overline{H}(x) \le f(\xi_0) + 2\varepsilon$$

when $|x - \xi_0| <$ some $\rho_\varepsilon$ $(\rho_\varepsilon \le \delta_\varepsilon)$. This concludes the proof. ∎

COROLLARY If there is a barrier at each boundary point, then

$$\underline{H} = \overline{H} \quad (\text{write } H = \overline{H} = \underline{H})$$

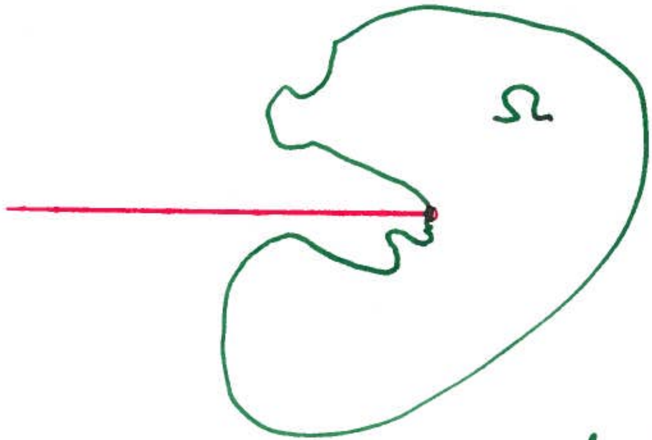
and

$$\lim_{x \to \xi} H(x) = f(\xi)$$

when $\xi \in \partial\Omega$.

**Example** $(n=2)$ $\Omega \subset B(0,1)$ = unit disk.

$0 \in \partial\Omega$, $(-\infty, 0] \cap \Omega = \phi$. [Exterior segment]. Then $0$ is a regular boundary point.



$$z = re^{i\theta}$$

$$-\pi < \theta < \pi$$

$$\omega(r,\theta) = \frac{\log\left(\frac{1}{r}\right)}{(\log r)^2 + \theta^2}$$

This is a barrier at $0$, when $\Omega$ is in the complement of $(-\infty, 0]$, $0 \in \partial\Omega$.

$$\Delta\omega = 0 \ ?$$

REMARK Every boundary point of a simply connected domain in the plane is regular

See Ahlfors.