

Normal families.

I mainly follow Ahlfors, so only the plan of the lecture is written.

1. $\Omega \subset \mathbb{C}$, \mathcal{F} - some family of functions $f: \Omega \rightarrow \mathbb{C}$
(not necessarily analytic)

2. Definition: \mathcal{F} is equicontinuous on $E \subset \Omega$

3. Example Analytic functions bounded on each compact subset in Ω

Notation $E \subset\subset \Omega$

4. Remind Uniform convergence on a set E

Notation: $f_n \Rightarrow f$ on E

5. Example: Power series

6. Fact: $\{f_n\}$ - analytic in Ω ;
 $f_n \Rightarrow f$ on each $E \subset\subset \Omega$ } \Rightarrow

$\Rightarrow f$ is analytic in Ω .

6ⁱⁱ $\{f_n\}$ - analytic, $f_n \Rightarrow f$ on compact sets \Rightarrow

7. Definition \mathcal{F} is a normal family - 2 -
in Ω .

Remark: The limit needs not belong to \mathcal{F} .



I skip the definition of the distance $d(a, b)$
and Theorems 9 and 10 as it is done
in Ahlfors book, implicitly we of course,
prove them.

8. Theorem (Arzela-Ascoli).

$E \subset \mathbb{C}$, \mathcal{F} -family of functions on E to \mathbb{C}
such that:

1. \mathcal{F} -equicontinuous

2. For some $M > 0$, $|f(z)| \leq M$ for all
 $z \in E$, $f \in \mathcal{F}$

Then each sequence of functions $\{f_n\} \subset \mathcal{F}$
contains a subsequence $\{f_{n_k}\}$ which converges
uniformly on E

I am not going to prove this. Do yourselves

~~to do this yourselves~~

14. Reminder: f is meromorphic in Ω if it is analytic in Ω except perhaps a countable set $\{z_n\} \subset \Omega$ where it has poles.

15. Meromorphic functions can be considered as functions: $f: \Omega \rightarrow \overline{\mathbb{C}}$ if replace S by $\overline{\mathbb{C}}$

condition

16. Now ~~the~~ condition of normality can be given for meromorphic functions.

Riemann mapping theorem

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1. Formulation.

2. Uniqueness

3. Univalent functions.

4. Hurwitz theorem

5. Limit of compact-wise convergent sequence of univalent functions is (if not a constant) also univalent.

6. Proof. Let $\Omega \subset \mathbb{D}$ be bounded & simply connected

$$z_0 \in \Omega$$

$$\mathcal{F} = \left\{ f; \text{univalent in } \Omega, f: \Omega \rightarrow \mathbb{D}, \right. \\ \left. f(z_0) = 0, f'(z_0) > 0 \right\}$$

A) $\mathcal{F} \neq \emptyset$

B) Take $g \in \mathcal{F}$ such that $g'(z_0) = \max \{ f'(z_0), f \in \mathcal{F} \}$

(Existence follows from normality)

$$g \neq \text{const since } g'(z_0) > 0$$

c) g -univalent $\Rightarrow g: \Omega \rightarrow g(\Omega) \subset \mathbb{D}$
 conformally

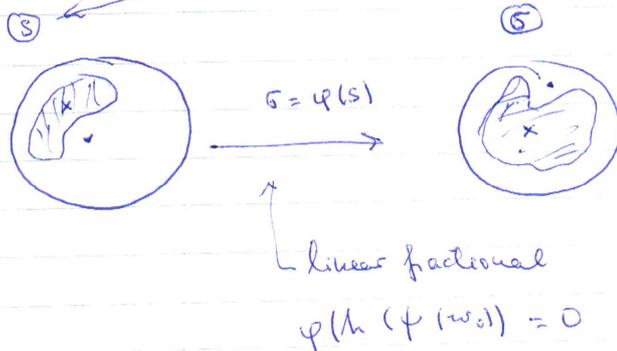
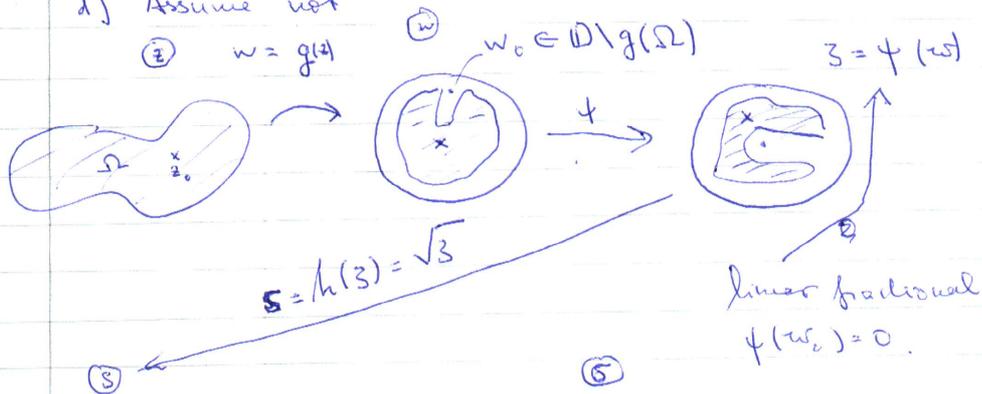
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We need: $g(\Omega) = \mathbb{D}$

d) Assume not

② $w = g(z)$

④ $w_0 \in \mathbb{D} \setminus g(\Omega)$



Denote $\Phi = \varphi \circ h \circ \varphi$: $\mathbb{D} \rightarrow \mathbb{D}$
 near $\varphi(w_0)$

e) This is expansion in hyperbolic metric because it is inverse to function $z \rightarrow z^2$ from \mathbb{D} onto \mathbb{D} .

φ, φ - are isometries in hyperbolic metric.

f) That is why

$$\frac{S(\Phi(w), 0)}{S(w, 0)} \rightarrow c > 1$$

as $w \rightarrow 0$

~~Now taking~~ On the other hand

$$\frac{S(\Phi(w), 0)}{S(w, 0)} \sim \left| \frac{\Phi(w)}{w} \right| \rightarrow |\Phi'(w)| = c > 1$$

Now taking $\Phi \circ g : \Omega \rightarrow \mathbb{D}$ we obtain contradiction.