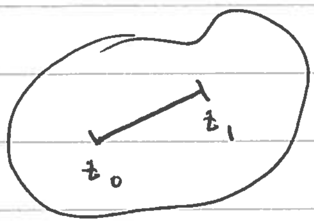
Erasing singularities

Theorem $\Omega \subset \mathbb{C}$ domain $[z_0, z_1] \subset \Omega$

$f \in \mathrm{Anal}(\Omega \setminus [z_0, z_1])$ and

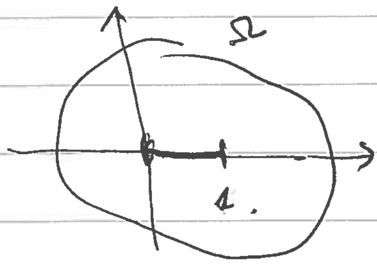
f is continuous in Ω



$\Rightarrow$ f is analytic in the whole Ω.

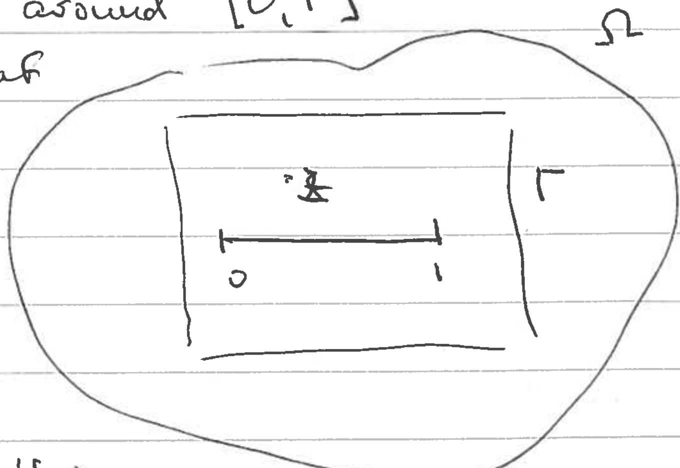
Remark We will see the segment can be replaced by any curve.

Proof: 1. For simplicity of notation $[z_0, z_1] = [0, 1]$



2. It suffices to prove analyticity at points $z \in [0, 1]$ only.

3. Take a rectangle Γ around $[0, 1]$
It suffices to prove that everywhere inside Γ

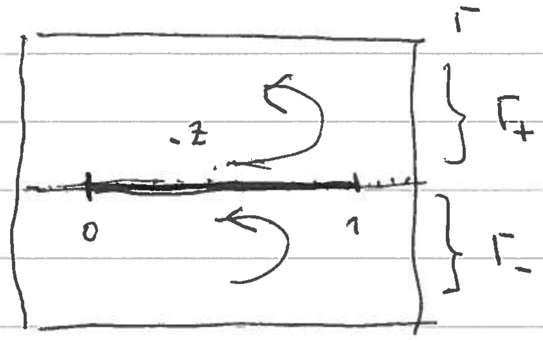


$$(*) \quad \underbrace{f(z) = \frac{1}{2i\pi} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta}$$

because this is analytic.

4. It suffice to prove (*) for all ~~z~~ inside Γ, $z \notin [0,1]$ Because both parts are continuous on $[0,1]$.

5.



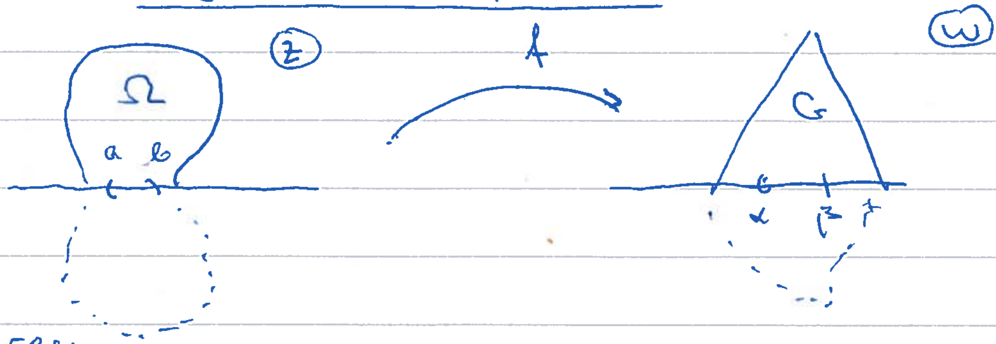
Take $z \notin [0,1]$, for example inside Γ_+

We have

$$\left.\begin{aligned} f(z) &= \frac{1}{2i\pi}\int_{\Gamma_+} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ 0 &= \frac{1}{2i\pi}\int_{\Gamma_-} \frac{f(\zeta)}{\zeta - z}\, d\zeta \end{aligned}\right\} +$$

$$f(z) = \frac{1}{2i\pi}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{— Done !}$$

Symmetry principle



Theorem

Given: $\Omega \subset \mathbb{C}_+$, $\partial\Omega \supset (a,b) \subset \mathbb{R}$

$f \in \text{Anal}(\Omega)$, $\operatorname{Im} f(x) = 0$ for $x \in (a,b)$

f - continuous in $\operatorname{clos}(\Omega)$

Then f - admits analytic prolongation to $\Omega \cup \bar{\Omega} \cup (a,b)$:

$$F(z) = \begin{cases} f(z), & z \in \Omega \\ f(x), & x \in (a,b) \\ \overline{f(\bar{z})}, & z \in \bar{\Omega} \end{cases}$$

Corollary ~~[illegible]~~: If f maps conformally

$f : \Omega \to G$, $G \subset \mathbb{C}_+$

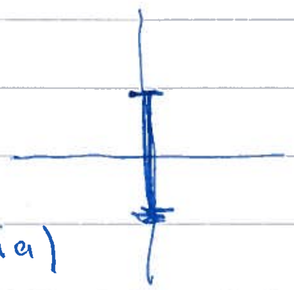
and $f(a,b) = (\alpha, \beta)$

Then

F maps conformally $\Omega \cup \bar{\Omega} \cup (a,b)$ onto

One can make conformal mappings by using symmetry principle.

<u>Example</u>



$\Omega = \mathbb{C} \setminus (-ia, ia)$

~~$G = \mathbb{C} \setminus (-i\alpha, i\alpha) \cup (-\beta, \beta)$~~

$G = \mathbb{C} \setminus \big((-i\alpha, i\alpha) \cup (-\beta, \beta)\big)$

<u>Exercise:</u>

Make assymetric case.

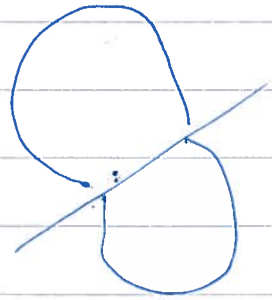
<u>Remark:</u> Consider these domains on extended plane

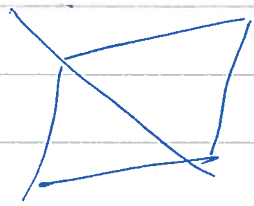
$\bar{\mathbb{C}} \setminus (-ia, ia)$; $\bar{\mathbb{C}} \setminus \big((-i\alpha, i\alpha) \cup (-\beta, \beta)\big)$

They are simply connected.

Remarks

• We can take symmetry with respect to any straight line

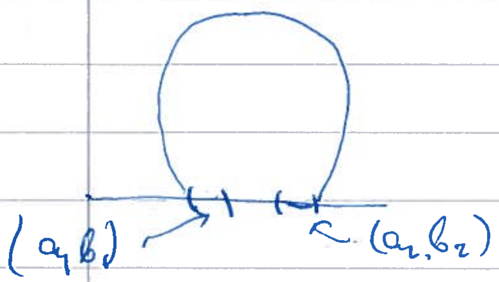




and even with respect to a circle

Exercise: How would you define symmetry with respect to a circle? (Hint Linear-fractional mapping).

• There could be two pieces of $\partial\Omega \cap \mathbb{R}$ where f is real



In this case $\Omega \cup \bar{\Omega} \cup (a_1 b_1) \cup (a_2 b_2)$ becomes multiconnected:

No prolongation through the middle part

Examples of conformal mappings

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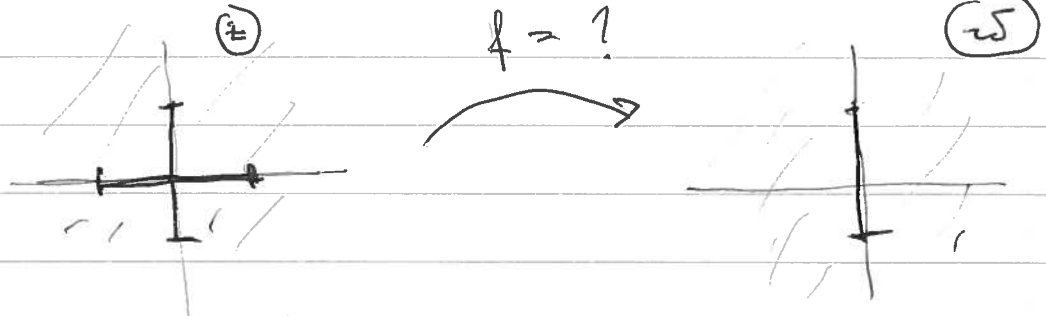
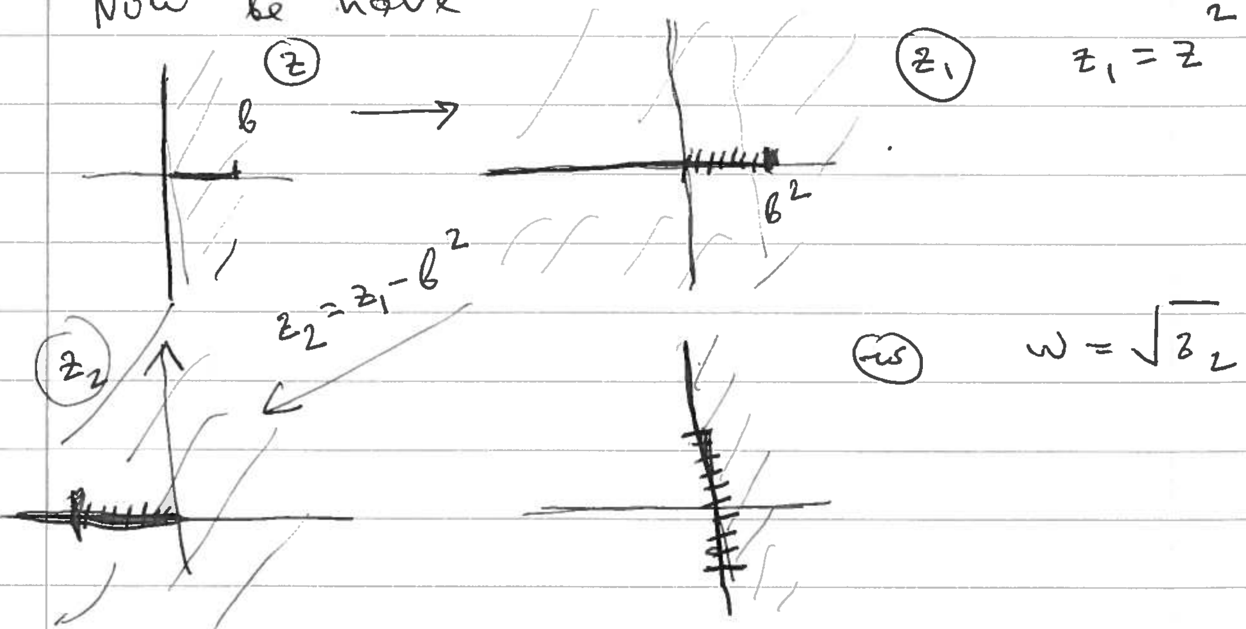


$\Omega = \mathbb{C} \setminus ((-i\alpha, i\alpha) \cup (-\beta, \beta))$ $\qquad$ $G = \mathbb{C} \setminus (-i\beta, i\beta)$

Solution We make mapping

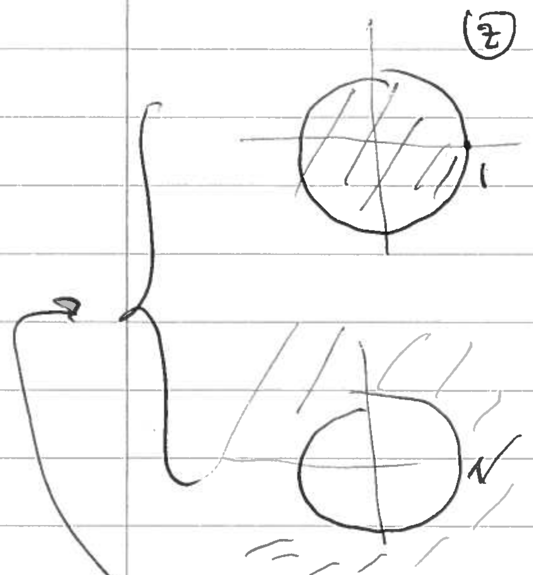
such that $(-i\beta, i\beta)$ be the image of the cut. Then apply the ~~reflection~~ symmetry principle.

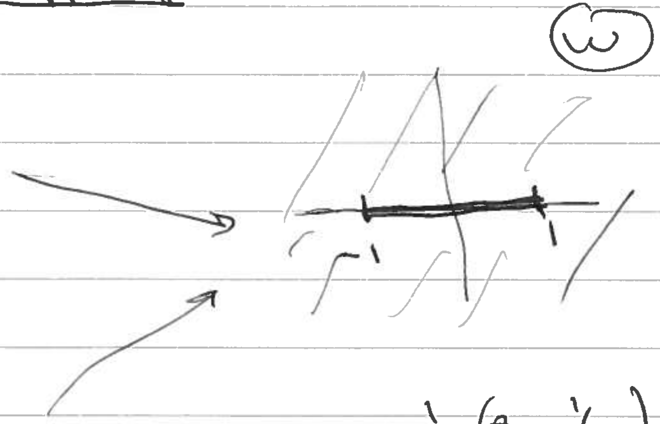
Now be have



By ++++++ I denote the image of the horizontal cut.

Zhukovskoi mapping





$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

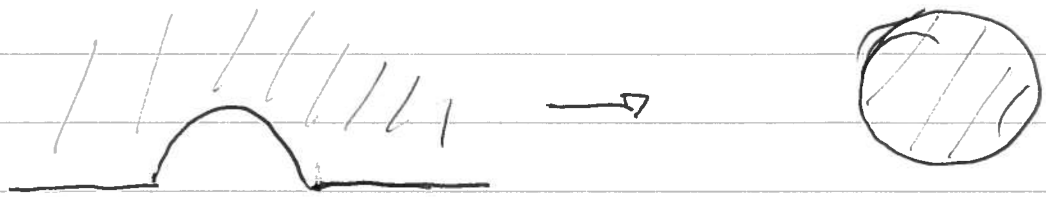
It maps conformally each of these domains onto the exterior of $[-1, 1]$

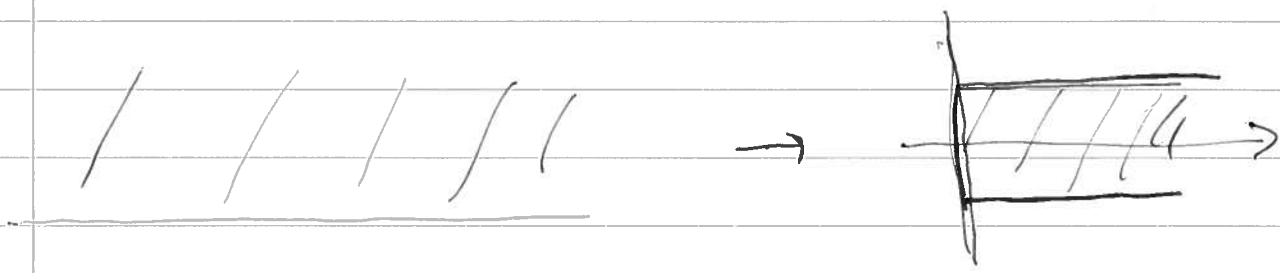
Inverse mapping: $z = w \pm \sqrt{w^2 - 1}$

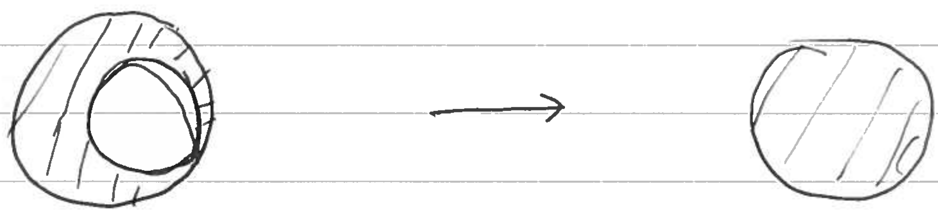
We have to discuss the following questions:

- Why $\sqrt{w^2-1}$ is well-defined outside $[-1,1]$?
- Two branches of $\sqrt{w^2-1}$. Which one to be chosen if we want to have mapping on $\mathbb{D}$ say?

More examples







Enjoy!