Applications of residus theorem.

1) Lagrange interpolation (see separate note).

2) Formula for inverse function:

$f : \Omega \to G$ - one-to-one, continuous up to ∂G;

$w \in G \Rightarrow f^{-1}(w) = \frac{1}{2i\pi} \int_{\partial\Omega} \frac{z f'(z)}{w - f(z)} dz$

3) Calculation of real integrals.

a. Reminder: ML theorem.

b. Main idea: make a closed curve (or boundary of some domain) then apply residus and make additional contribution neglegible.

c. Rational function.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

d. Simple trigonometric functions

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx$$

e. Integrals with branching points

$$\int_0^\infty \frac{x^\alpha}{(1+x)^2}\,dx = \frac{\pi\alpha}{\sin\pi\alpha} \qquad -1<\alpha<1$$

keyhall contour

$$\int_0^1 \frac{x^{1-p}(1-x)^p}{(1+x)^2}\,dx, \quad -1<p<2$$

f. Integrals of rational functions

- $\int_0^{2\pi} R(\cos t, \sin t)\,dt$ - general pattern

  ↑ rational function

- Example (from exam in Matte 4, Fall 2016)

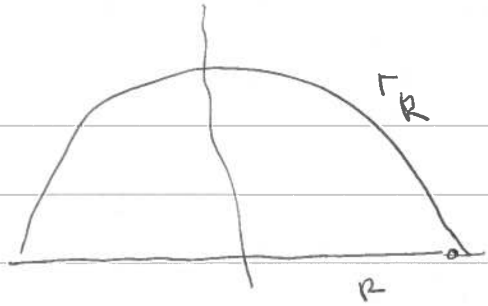
$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} =$$

4) Jordan lemma:

Example: $I = \int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} \underbrace{\frac{x e^{ix}}{1+x^2}}_{f(x)}\,dx$

We have

$$f(z) = \frac{z}{z^2+1} e^{iz}$$



Inequality:

$$z \in \Gamma_R \Rightarrow |f(z)| \leq \max_{|z|=R} \left|\frac{z}{z^2+1}\right| \cdot \max_{|z|=R} |e^{iz}| \sim \frac{1}{R}$$

Length of $\Gamma_R \sim R$

We cannot apply ML-theorem!

<u>Lemma (Jordan)</u>

<u>$a>0$</u>

$$\int_{\Gamma_R} |e^{iaz}|\,|dz| < \text{Const}, \quad \text{for all } R$$

independent of R

<u>Corollary</u>:

$$\varphi(z) \to 0 \quad \text{as} \quad z \to \infty \quad \Rightarrow$$

$$\Rightarrow \int_{\Gamma_R} e^{iaz} \varphi(z)\,dz \to 0 \quad \text{as} \quad R \to \infty.$$

Now we can complete evaluation of integral I since we have in this example

$$\varphi(z) = \frac{z}{z^2+1} \to 0 \quad \text{as } z \to \infty.$$

5) Integral in principal value (beginning)

Example:

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx \qquad (*)$$

Test: why this converges?

We have:

$$I = \underbrace{\mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx}$$

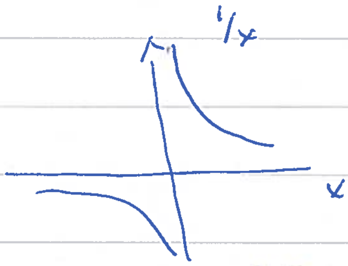
Does this ~~makes sense~~ !!

We have non-integrable singularity at $x=0$.

But I as written in $(*)$ is perfectly well defined. So $\underbrace{\quad}$ SHOULD make some sense

WHICH ?

Bright idea:



near zero negative and positive parts should kill each other!

Claim: There exists

$$\lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} \frac{e^{ix}}{x}\,dx \qquad (1)$$

Notation
$$\text{V.P.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = \underbrace{\lim_{\varepsilon \downarrow 0} \int_{|x|>\varepsilon} \frac{e^{ix}}{x}\,dx}_{\text{Integral in principal value.}}$$

Proof:

$$\int_{|x|>\varepsilon} \frac{e^{ix}}{x}\,dx = \underbrace{\int_{|x|>1} \frac{e^{ix}}{x}\,dx}_{} + \int_{\varepsilon<|x|<1} \frac{e^{ix}}{x}\,dx$$

This exists and does not depend on ε.

The second summand:

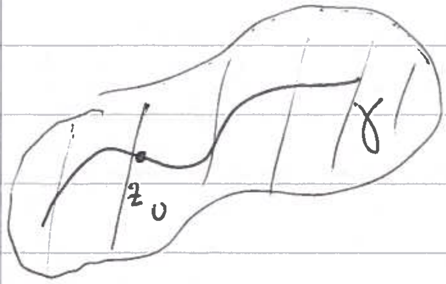
$$\int_{\varepsilon<|x|<1} \frac{e^{ix}}{x}\,dx = \underbrace{\int_{\varepsilon<|x|<1} \frac{e^{ix}-1}{x}\,dx}_{} + \underbrace{\int_{\varepsilon<|x|<1} \frac{dx}{x}}_{}$$

This ~~has~~ approaches a limit since function is integrable

This is 0 because $1/x$ is an odd function

Remark: If we did not take symmetric vicinity nothing would work.

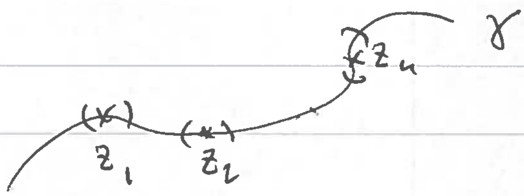
6) Integral in principal value (general definition and theorem)



γ - smooth curve
f - analytic in a vicinity of γ except $z_0 \in \gamma$ where it has a simple pole

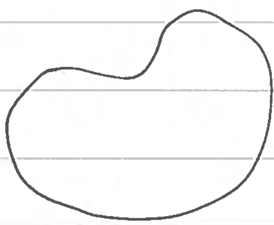
$$\text{V.P.} \int_\gamma f(z)\,dz := \lim_{\varepsilon \to 0} \int_{\substack{z \in \gamma \\ |z - z_0| > \varepsilon}} f(z)\,dz$$

If there are several simple poles
Then



$$\text{V.P.} \int_\gamma f(z)\,dz := $$
$$= \lim_{\varepsilon \to 0} \int_{\substack{z \in \gamma \\ |z - z_j| > \varepsilon}} f(z)\,dz$$

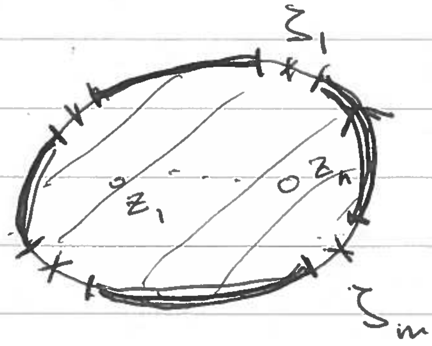
Theorem Ω - bounded, $\partial\Omega$ smooth, f is analytic in a vicinity of Ω except $z_1, z_2, \ldots, z_n \in \Omega$ and also $\zeta_1, \ldots \zeta_m \in \partial\Omega$, where it has simple poles



Then

$$\frac{1}{2\pi i} \text{V.P.} \int_{\partial\Omega} f(z)\,dz = \sum_1^n \operatorname{Res}_{z_j} f + \frac{1}{2} \sum_1^m \operatorname{Res}_{\zeta_k} f$$

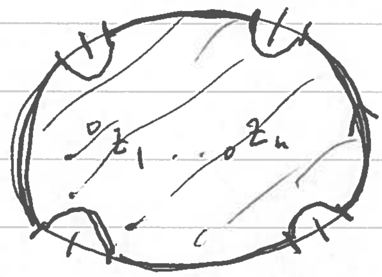
Proof



$$\gamma_\varepsilon = \{ z \in \partial\Omega ; \ \operatorname{dist}(z, \{\zeta_j\}_1^m) > \varepsilon \}$$

$$VP \frac{1}{2i\pi} \int_{\partial\Omega} f(z)\, dz = \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{\gamma_\varepsilon} f(z)\, dz .$$

Auxilliary domain



$$\Omega_\varepsilon = \{ z \in \Omega, \operatorname{dist}(z, \{\zeta_j\}) > \varepsilon \}$$

~~$\partial\Omega_\varepsilon$ ... $C_\varepsilon(k)$~~

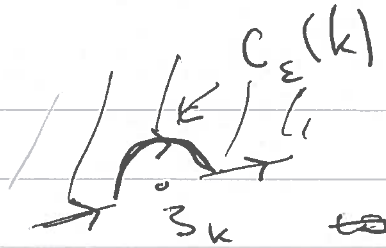
$$\partial\Omega_\varepsilon = \gamma_\varepsilon \cup \left(\bigcup_1^m C_\varepsilon(k) \right), \quad C_\varepsilon(k) = \{ z \in \Omega : |z - \zeta_k| = \varepsilon \}$$

Points $\{\zeta_k\}$ are not in Ω_ε

$$\Rightarrow \frac{1}{2i\pi} \int_{\partial\Omega_\varepsilon} f(z)\, dz = \sum_{l=1}^{n} \operatorname{Res}_{z_l} f \qquad (*)$$

residus at inner points.

Bigger scale:



clockwise (negative) direction on $C_\varepsilon(k)$

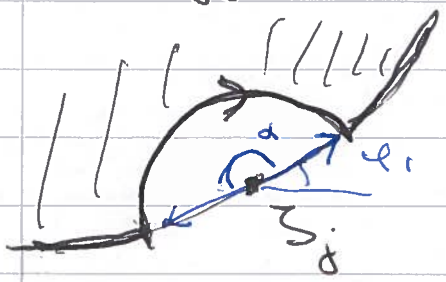
$$\frac{1}{2\pi i}\int_{\partial\Omega_\varepsilon} = \left(\frac{1}{2\pi i}\int_{\gamma_\varepsilon} - \sum_{j=1}^{m}\frac{1}{2\pi i}\int_{C_\varepsilon(j)}\right) f(z)\,dz \qquad (*) \quad \Rightarrow$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma_\varepsilon} f(z)\,dz = \sum_{j=1}^{m}\frac{1}{2\pi i}\int_{C_\varepsilon(j)} f(z)\,dz + \underbrace{\sum_{k=1}^{n}\operatorname{Res}_{z_k} f}_{\text{independent of } \varepsilon}$$

We need to prove that, for each j

$$\frac{1}{2\pi i}\int_{C_\varepsilon(j)} f(z)\,dz \to \frac{1}{2}\operatorname{Res}_{z_j} f$$

Bigger scale



$\alpha \to \pi$ as $\varepsilon \to 0$ because the curve is smooth

Let $a_j = \mathrm{Res}_{\zeta_j} f$

$$f(z) = \frac{a_j}{z-\zeta_j} + \{\text{regular part}\}.$$

We have $\frac{1}{2i\pi} \int_{C_\varepsilon(j)} \{\text{regular part}\}\, dz \to 0$ as $\varepsilon \to 0$

because $\{\text{regular part}\}$ is ~~bounded~~ bounded.

$$\frac{1}{2i\pi} \int_{C_\varepsilon(j)} \frac{a_j}{z-\zeta_j}\, dz = a_j \frac{\alpha}{2\pi} \to \frac{1}{2} a_j$$

because $\alpha \to \pi$

use parametrization

This is what we need

$$\underbrace{\lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_{\gamma_\varepsilon} f(z)\, dz}_{\text{"} \mathrm{v.p.} \frac{1}{2i\pi} \int_{\partial\Omega} f(z)\, dz} \to \frac{1}{2} \sum \mathrm{Res}_{\zeta_j} f + \sum \mathrm{Res}_{z_k} f$$

7. Now we can complete calculation of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx$$