Including poles into the consideration

Reminder
- pole of order n
- meromorphic function in Ω

Facts: • Poles are separated inside Ω, but they may accumulate to the boundary } Prove this yourselves.

- Factorization: f has a finite # of poles and zeros in Ω.

Respectively zeros $z_1, z_2, \dots, z_m$ of multiplicities $k_1, k_2, \dots k_m$

poles at $\zeta_1, \dots, \zeta_n$ of order $l_1, \dots, l_n$

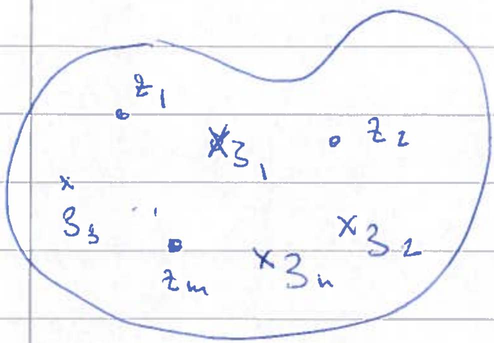
Then $$f(z) = (z-z_1)^{k_1} \cdots (z-z_m)^{k_m} (z-\zeta_1)^{-l_1} \cdots (z-\zeta_n)^{-l_n} g(z)$$

g - has no zeros no poles.

~~Theorem~~

Theorem: Ω - bounded; f - meromorphic in Ω, f, f' continuous up to $\partial\Omega$. $f|_{\partial\Omega} \neq 0$.

$z_1, \dots, z_m, \zeta_1, \dots, \zeta_n$ - zeros and poles of f of orders $k_1, \dots, k_m, l_1, \dots, l_n$



Then:

$$\underbrace{k_1 + \dots + k_m}_{N_0} \underbrace{- l_1 - \dots - l_n}_{N_\infty} = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

Definition: Order of a point:

$$n_f(z_0) = \begin{cases} n & , z_0 \text{ is zero of order } n \\ -n & , z_0 \text{ is pole of order } n \\ 0 & f(z_0) \neq 0, \ f\text{-analytic at } z_0 \end{cases}$$

The previous result:

$$N_0 - N_\infty = \frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz$$

Test question

$$\boxed{\text{Do you recognize what does } f'/f \text{ mean?}}$$

Reminder: f is analytic in Ω

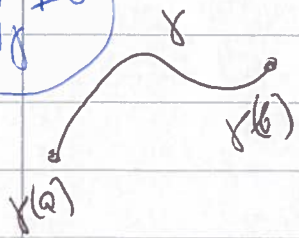
- $\log f(z)$ is not defined uniquely.
- $\log f(z)$ is defined up to constant.
- What happens with $\log f(z)$ when f goes around zero.

$$\frac{f'(z)}{f(z)} = \log f'(z) \text{ — is defined uniquely!}$$

Definition:

$$\int_\gamma d\log|f(z)| = \log|f(\gamma(b))| - \log|f(\gamma(a))|$$

$f|_\gamma \neq 0$!



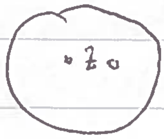
$$\int_\gamma d\arg f(z) = \underbrace{\arg f(\gamma(b)) - \arg f(\gamma(a))}_{\Delta_\gamma \arg f}$$

depends upon γ.

Example:

$f(z)=(z-z_0)^n$ $\qquad \Delta_\gamma \arg f = n$

γ



Previous theorem (Argument principle).

Ω - bounded, f - meromorphic ~~on~~ in Ω continuous up to $\partial\Omega$, $f|_{\partial\Omega} \neq 0$

$$\Rightarrow \frac{1}{2\pi} \Delta_{\partial\Omega} \arg f = N_0 - N_\infty$$

Remark: here we need not demand f' extends to $\partial\Omega$!

Rouché theorem:

Ω - bounded, $f, h \in \mathrm{Anal}(\Omega)$, continuous in $\mathrm{Clos}(\Omega)$

and $|f(z)| > |h(z)|$ for all $z \in \partial\Omega$

Then: $f(z)$ and $f(z)+h(z)$ have the same # of zeros with account multiplicities.

Proof: $$f(z)+h(z) = f(z)\underbrace{\left[1+\frac{h(z)}{f(z)}\right]}_{\text{in the right half plane as } z \in \partial\Omega}$$

$$\Rightarrow \arg[f(z)+h(z)] = \arg f(z) + \underbrace{\arg(1+h(z)/f(z))}_{\text{increment of this on } \partial\Omega \text{ is } 0}$$

$\Longrightarrow$ $\boxtimes$

<u>Example</u> Another proof of the fundamental theorem of algebra.

* * *

<u>Definition</u> f - analytic near z_0

z_0 is <u>regular point</u> if $f'(z_0) \neq 0$

<u>critical point</u> if $f'(z_0) = 0$.

We will study the behavior of f at the regular point first.