Local behavior (continuation)

Remind the theorem we are proving:

z_0 - isolated singularity of f

$$\begin{cases} \cdot \quad f \text{ is bounded near } z_0 \iff \text{removable} \\ \cdot\cdot \quad f(z) \to \infty \text{ as } z \to z_0 \iff \text{pole} \\ \therefore \quad \text{No } \lim\limits_{z \to z_0} f(z) \text{ exists (finite or infinite)} \\ \qquad \iff \text{essential singularity} \end{cases}$$

• Done (at the previous lecture)

•• $\Leftarrow$ Done at the previous lecture.

Proving •• $\Rightarrow$

~~Consider~~

We know $f(z) \to \infty$, $z \to z_0$

Consider $g(z) = 1/f(z) \to 0$, $z \to z_0$

• $\Rightarrow$ z_0 - removable for $g(z)$,
after removing $g(z_0) = 0$

$$\Rightarrow g(z) = \sum_{n=N}^{\infty} c_n (z - z_0)^n, \quad c_N \neq 0$$

$$\Rightarrow g(z) = (z - z_0)^N \underbrace{\left[c_N + c_{N+1} (z - z_0) + \cdots \right]}_{\varphi(z)}$$

$\varphi(z_0) \neq 0 \Rightarrow \dfrac{1}{\varphi(z)}$ analytic at z_0,

$$\frac{1}{\varphi(z)} = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Now

$$f(z) = \frac{1}{g(z)} = \frac{1}{(z - z_0)^N} \frac{1}{\varphi(z)} =$$

$$= \underbrace{\sum_{k=0}^{N-1} a_k (z - z_0)^{k-N}}_{\text{finite principal part}} + \underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^{k-N}}_{\text{regular par}}$$

↑ pole ∎

Proving ∴ No need to prove! All other options are eliminated.

∎

Important corollary (Factorization)

\# $\Omega \subset \mathbb{C}$; $f : \Omega \to \mathbb{C}$ - analytic and there are $z_1, \dots, z_n \in \Omega$ such that $f(z_j) = 0$, $j = 1, 2, \dots, n$.

Then:

$$f(z) = (z - z_1) \cdots (z - z_n) \underbrace{g(z)}_{\text{analytic in } \Omega}.$$

\# z_0 - zero of $f \iff z_0$ - pole of $1/f(z)$

\# By the way All zeros (poles) of an analytic function are separated and may have limit points only on the boundary of domain.

Theorem (Weierstrass)

$\{ z_0$ - essential singularity of f, $\delta > 0 \} \Longrightarrow$

$\Longrightarrow f(\{z : 0 < |z - z_0| < \delta\})$ is dense in $\mathbb{C}$.

Proof: Assume not i.e.

for some $w_0 \in \mathbb{C}$, $\varepsilon > 0$ $|f(z) - w_0| > \varepsilon$
for all $z : |z - z_0| < \delta$.

Consider $\varphi(z) = \dfrac{1}{f(z) - w_0}$ - bounded around z_0

$\Longrightarrow z_0$ - removable singularity of φ.

$$\varphi(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

$$f(z) = \frac{1}{\varphi(z)} + w_0$$

Two options: a) $\varphi(z_0) \neq 0 \Longrightarrow f$ is bounded near z_0
$\Longrightarrow$ removable singularity

b) $\varphi(z_0) = 0 \Longrightarrow \varphi(z) \to 0$, $z \to z_0 \Longrightarrow$
$\Longrightarrow f(z) \to \infty$, $z \to z_0$
$\Bigg\} \Longrightarrow$ pole!

Contradiction!

~~Mainly~~ This and previous lecture mainly Ch. 4 Sec. 3.1, 3.2 in Ahlfors' book.

Related definitions / notions

- Singularity at ∞. Examples.
- Algebraic order of a function at a point.
- Meromorphic functions.

~~Topics to be~~

Topics from Matte 4K to be refreshed for the next lecture:

- Curves
- Integrals along the curves
- Cauchy theorem
- Cauchy representation.

(I will remind them, but very briefly).

Correspondence between lectures and Ahlfors book:

The main results can be found in Ch. 4, Sections 3.1 and 3.2