Compactness

1. (X, d) Cauchy sequence: $\{x_n\} \subset X$:
$d(x_n, x_m) \to 0$, $n, m \to \infty$.

2. Complete metric space. Each C.s. has a limit.

3. Notion of subspace. $Y \subset X$, (Y, d)

4. $Y \subset X$, (Y, d) - complete $\Leftrightarrow$ Y is closed.

5. Remind about continuous functions on a segment.

6. Various definitions of compactness.

- X is compact if any open covering contains a finite subcovering.

7. Claim: X is a compact set $\Rightarrow$ X is complete.

8. Inverse is wrong: $\mathbb{R}$

9. Boundedness X is bounded if $d(x, y) < M$; $\forall x, y \in X$

10. Claim: X - compact set $\Rightarrow$ X is bounded.

11. What about inverse statement?
(I postpone the answer).

12. Total boundedness:
$$\forall \varepsilon > 0 \ \exists \{x_j\}_1^N \subset X \text{ such that } X \subset \cup B_\varepsilon(x_j)$$
finite!

Terminology: $\{x_j\}_1^N$ - ε-net

13. Remark: Subset of totally bounded is totally bounded as well.

14. ~~E~~ Claim: X is compact $\Leftrightarrow$ X is complete and totally bounded.

- Proof $\Rightarrow$: straightforward.
- Proof $\Leftarrow$: Assume wrong and make a Cauchy sequence.

15. Corollary: $X \subset \mathbb{R}^n$.
X - compact $\Leftrightarrow$ X closed and bounded.

16. ~~Ex~~. This is wrong in more general setting:
Example: Qube in ℓ^2

18. Limit point of $\{x_n\}_1^\infty$.

y is a limit point $\Leftrightarrow \exists \{x_{n_k}\} \subset \{x_n\}$:

$$x_{n_k} \to y$$

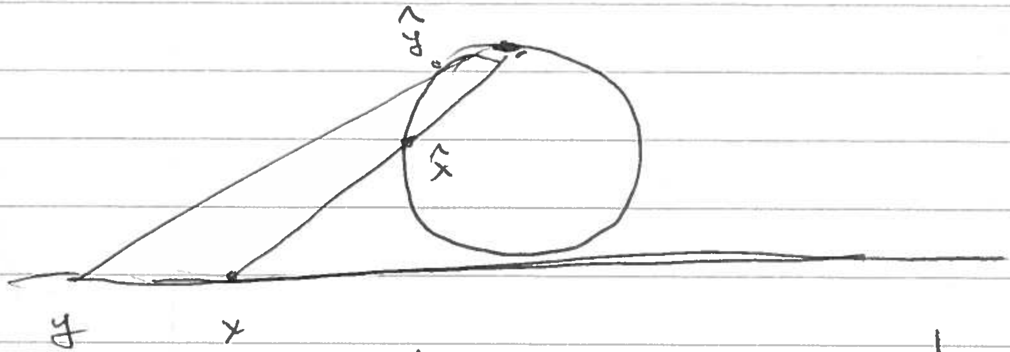
Equivalently: $\forall \varepsilon > 0 \quad \#(B_\varepsilon(y) \cap \{x_n\}) = \infty$.

19. Claim: X is compact $\Leftrightarrow$ any infinite sequence has a limit point.

20. Resume: 6, 14, and 19 ~~are~~

can be taken as equivalent definitions of compactness in metric spaces.

21. How one introduces metric in $\mathbb{R}$ in order to make it a compact set?



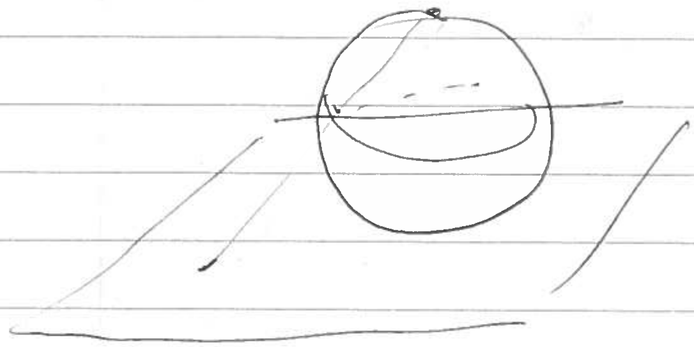
We add <u>infinity</u>! $\mathbb{R} \cup \{\infty\} = \overline{\mathbb{R}}$

$x, y \in \mathbb{R}$, new distance $\tilde{d}(x,y) = d(\hat{x},\hat{y})$

Euclidean distance on the circle

With this metric $\overline{\mathbb{R}}$ is a compact set.

22. <u>Stereographic projection</u> The same for $\mathbb{C}$:



$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

Special name:

<u>Riemann sphere</u>

Homework:

1. Let (X, d) be a metric space. Verify the following properties of open and closed sets:

   - if $\{G_n\}_{n=1}^{\infty}$, $G_n \subset X$ are open sets then $\bigcup_1^{\infty} G_n$ is open as well and, for each $N < \infty$ $\bigcap_1^{N} G_n$ is open as well.

   - if $\{F_n\}_{n=1}^{\infty}$, $F_n \subset X$ are closed sets then $\bigcap_1^{\infty} F_n$ is closed as well and, for each $N < \infty$ $\bigcup_1^{N} F_n$ is closed.

2. Give examples that one cannot take $N = \infty$ in problem 1.

Problems from Ahlfors' book: Chapter 3

Sect. 1.2 : 4, 7

Sect 1.3 1-4

Sect 1.4 2-5