4. Show that the circumference of a hyperbolic disk of radius ρ is $2\pi \sinh \rho$. *Hint.* Show first that the hyperbolic circumference of a Euclidean disk of radius r centered at 0 is $4\pi r/(1-r^2)$.

5. We define the **hyperbolic area** of a subset E of $\mathbb{D}$ to be

$$4 \iint_E \frac{dx\, dy}{(1-|z|^2)^2}.$$

Show that the hyperbolic area is invariant under conformal self-maps of $\mathbb{D}$. Show that the hyperbolic area of a hyperbolic disk of radius ρ is given by

$$2\pi(\cosh \rho \ - \ 1) \ = \ \pi\rho^2 + \frac{\pi}{12}\rho^4 + \mathcal{O}(\rho^6).$$

6. Establish the following, for the spherical metric. (a) The circumference of a spherical disk of radius ρ is $2\pi \sin \rho$, $0 < \rho < \pi$. (b) The area of a spherical disk of radius ρ is given by

$$2\pi(1 - \cos \rho) \ = \ \pi\rho^2 - \frac{\pi}{12}\rho^4 + \mathcal{O}(\rho^6).$$

(c) The geodesics in the spherical metric correspond to great circles on the sphere. *Hint.* It suffices to show that the shortest curve from 0 to ε in the spherical metric is the straight line segment joining them.

7. Show that an isometry of the hyperbolic disk $\mathbb{D}$ is either a conformal self-map of $\mathbb{D}$ or the composition of a conformal self-map and the reflection $z \mapsto \bar{z}$.

8. Let $f(z) = (az+b)/(cz+d)$, where $ad-bc = 1$. Show that $f(z)$ is an isometry in the spherical metric if and only if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary.

9. Show that the function $f(z) = z^2$ is strictly contracting with respect to the hyperbolic metric on any subdisk $\{|z| \le r\}$, $0 < r < 1$, and that any branch of the square root function is strictly expanding, by establishing the following. (a) For fixed r, $0 < r < 1$, show that

$$\rho(z^2, \zeta^2) \ \le \ \frac{2r}{1+r^2}\rho(z,\zeta), \qquad |z|, |\zeta| \le r.$$

When does equality hold? (b) Show that the constant $2r/(1+r^2)$ in (a) is sharp. (c) For fixed s, $0 < s < 1$, show that

$$\rho\left(\pm\sqrt{z}, \pm\sqrt{\zeta}\right) \ \ge \ \frac{1+s}{2\sqrt{s}}\rho(z,\zeta), \qquad |z|, |\zeta| \le s.$$

10. Show that

$$d(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|, \qquad |z|,|w|<1,$$

satisfies the triangle inequality, that is, $d(z,w) \le d(z,\zeta)+d(\zeta,w)$ for all $z,\zeta,w \in \mathbb{D}$. *Remark.* This can be regarded as the analogue of the chordal metric for the sphere (defined in Section I.3). Except for the constant factor 2, the hyperbolic metric is the infinitesimal version of the metric function $d(z,w)$.

11. Show that the metric function $d(z,w)$ defined in the preceding exercise satisfies

$$d(f(z),f(w)) \le d(z,w), \qquad |z|,|w|<1,$$

for any analytic function $f(z)$ from $\mathbb{D}$ to $\mathbb{D}$. Show that equality obtains whenever $f(z)$ is a conformal self-map of $\mathbb{D}$, and otherwise there is strict inequality for all $z \ne w$.

12. A conformal map $g(z)$ of a domain D onto the open unit disk $\mathbb{D}$ induces the metric ρ_D on D defined by

$$d\rho_D(z) = \frac{2|g'(z)|}{1-|g(z)|^2}|dz|, \qquad z \in D.$$

Show that ρ_D is independent of the conformal map $g(z)$ of D onto $\mathbb{D}$. *Remark.* The metric ρ_D is called the **hyperbolic metric** of the simply connected domain D.

13. Show that the hyperbolic metric of the upper half-plane $\mathbb{H}$ is given by

$$d\rho_{\mathbb{H}}(z) = \frac{|dz|}{y}, \qquad z = x+iy,\ y>0.$$

What are the geodesics in the hyperbolic metric? Illustrate with a sketch.

14. Show that the horizontal strip $S = \{-\pi/2 < \operatorname{Im} z < \pi/2\}$ has hyperbolic metric

$$d\rho_S(z) = \frac{|dz|}{\cos y}, \qquad z = x+iy,\ -\pi/2 < y < \pi/2.$$

Sketch the hyperbolic geodesics that are orthogonal to the vertical interval $\{iy : -\pi/2 < y < \pi/2\}$.

15. The **curvature** of the metric $\sigma(z)|dz|$ is defined to be

$$\kappa(z) = -\frac{1}{\sigma(z)^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\sigma(z).$$

Find the curvature of each of the spherical, the hyperbolic, and the Euclidean metrics.

16. (Wolff-Denjoy Theorem.) Let $f(z)$ be an analytic function from $\mathbb{D}$ to $\mathbb{D}$. Let $f_n(z)$ denote the nth iterate of $f(z)$, and let K_r denote the closed disk $\{|z| \leq r\}$.
    (a) Show that if $f(z)$ is not a conformal self-map of $\mathbb{D}$, then for any $r < 1$ there is a constant $c < 1$ such that $\rho(f(z), f(w)) \leq c\rho(z, w)$ for $z, w \in K_r$.
    (b) Show that if the image $f(\mathbb{D})$ is contained in K_r for some $r < 1$, then the iterates $f_n(z)$ converge uniformly on $\mathbb{D}$ to a fixed point for $f(z)$.
    (c) Show that if $f(z)$ is not a conformal self-map of $\mathbb{D}$, and if there is $r < 1$ such that the iterates of some point $z_0 \in \mathbb{D}$ visit K_r infinitely often, then the iterates $f_n(z)$ converge normally on $\mathbb{D}$ to a fixed point of $f(z)$. *Hint.* First find the fixed point.
    (d) Show that if the iterates of some point $z_0 \in \mathbb{D}$ tend to the unit circle $\partial\mathbb{D}$, then there is a point $\zeta \in \partial\mathbb{D}$ (the **Wolff-Denjoy point**) such that the iterates $f_n(z)$ converge normally on $\mathbb{D}$ to ζ. *Hint.* Suppose $z_0 = 0$. Define $g_\varepsilon(z) = (1-\varepsilon)f(z)$, let z_ε be the fixed point of $g_\varepsilon(z)$, and let D_ε be the hyperbolic disk centered at z_ε with 0 on its boundary. Show that the limit D of the D_ε's is a Euclidean disk that is invariant under $f(z)$ and whose boundary meets $\partial\mathbb{D}$ in exactly one point.