Exercises for IX.2

1. A **finite Blaschke product** is a rational function of the form

$$B(z) = e^{i\varphi}\left(\frac{z-a_1}{1-\bar{a}_1 z}\right)\cdots\left(\frac{z-a_n}{1-\bar{a}_n z}\right),$$

where $a_1, \ldots, a_n \in \mathbb{D}$ and $0 \le \varphi \le 2\pi$. Show that if $f(z)$ is continuous for $|z| \le 1$ and analytic for $|z| < 1$, and if $|f(z)| = 1$ for $|z| = 1$, then $f(z)$ is a finite Blaschke product.

2. Show that $f(z) = (3+z^2)/(1+3z^2)$ is a finite Blaschke product.

3. Suppose $f(z)$ is analytic for $|z| < 3$. If $|f(z)| \le 1$, and $f(\pm i) = f(\pm 1) = 0$, what is the maximum value of $|f(0)|$? For which functions is the maximum attained?

4. For fixed $z_0, z_1 \in \mathbb{D}$, find the maximum value of $|f(z_1) - f(z_0)|$ among all analytic functions $f(z)$ on the open unit disk $\mathbb{D}$ satisfying $|f(z)| < 1$. Determine for which such functions the maximum value is attained. *Hint.* Consider first the case where $z_0 = r > 0$ and $z_1 = -r$, and show that the maximum is $2r$, attained only for $f(z) = \lambda z$, $|\lambda| = 1$.

5. Show that any conformal self-map of the upper half-plane has the form

$$f(z) = \frac{az+b}{cz+d}, \qquad \operatorname{Im} z > 0,$$

where a, b, c, d are real numbers satisfying $ad - bc = 1$. When do two such coefficient choices for a, b, c, d determine the same conformal self-map of the upper half-plane?

6. Show that the conformal maps of the upper half-plane onto the open unit disk are of the form

$$f(z) = e^{i\varphi}\frac{z-a}{z-\bar{a}}, \qquad \operatorname{Im} a > 0,\ 0 \le \varphi \le 2\pi.$$

Show that a and $e^{i\varphi}$ are uniquely determined by the conformal map.

7. Show that every conformal self-map of the complex plane $\mathbb{C}$ has the form $f(z) = az + b$, where $a \ne 0$. *Hint.* The isolated singularity of $f(z)$ at ∞ must be a simple pole.

8. Show that every conformal self-map of the Riemann sphere $\mathbb{C}^*$ is given by a fractional linear transformation.

9. Show that any conformal self-map of the punctured unit disk $\{0 < |z| < 1\}$ is a rotation $z \mapsto e^{i\varphi} z$.

10. Show that any conformal self-map of the punctured complex plane $\{0 < |z| < \infty\}$ is either a multiplication $z \mapsto az$, or such a multiplication followed by the inversion $z \mapsto 1/z$.

11. Let $D = \mathbb{C}\backslash\{a_1, \dots, a_m\}$ be the complex plane with m punctures. Show that any conformal self-map of D is a fractional linear transformation that permutes $\{a_1, \dots, a_m, \infty\}$.

12. Determine the conformal self-maps of the following domains D: (a) $D = \mathbb{C}\backslash\{0,1\}$, (b) $D = \mathbb{C}\backslash\{-1,0,1\}$, (c) $D = \mathbb{C}\backslash\{-1,0,2\}$.

13. Suppose $f(z)$ is an analytic function from the open unit disk $\mathbb{D}$ to itself that is not the identity map z. Show that $f(z)$ has at most one fixed point in $\mathbb{D}$. *Hint.* Make a change of variable with a conformal self-map of $\mathbb{D}$ to place the fixed point at 0.

14. Suppose $f(z)$ is an analytic function from the open unit disk $\mathbb{D}$ to itself that is not a conformal self-map, and denote by $f_n(z)$ the nth iterate of $f(z)$. Show that if $f(z)$ has a fixed point $z_0 \in \mathbb{D}$, then $f_n(z)$ converges to z_0 for each $z \in \mathbb{D}$. Show that for each $r < 1$, the convergence is uniform for $|z| \le r$. *Hint.* See Exercise 1.8.

15. We say that two conformal self-maps f and g of $\mathbb{D}$ are **conjugate** if there is a conformal self-map h of $\mathbb{D}$ such that $g = h \circ f \circ h^{-1}$. (See the exercises for Section II.7.) Let f be a conformal self-map of $\mathbb{D}$ that is not the identity map z. (a) Show that either f has two fixed points on $\partial\mathbb{D}$, counting multiplicity, or f has one fixed point in $\mathbb{D}$. (b) Show that f has a fixed point in $\mathbb{D}$ if and only if f is conjugate to a rotation $g(z) = e^{i\varphi}z$. (c) Show that rotations by different angles are not conjugate. (d) Show that f has two distinct fixed points on $\partial\mathbb{D}$ if and only if f is conjugate to $g(z) = (z-s)/(1-sz)$ for some s satisfying $0 < s < 1$. (e) Show that g's for different s's are not conjugate. (f) Show that any two conformal self-maps of $\mathbb{D}$ with one fixed point on $\partial\mathbb{D}$ (of multiplicity two) are conjugate.

3. Hyperbolic Geometry

Suppose $w = f(z)$ is a conformal self-map of the open unit disk $\mathbb{D}$. From Pick's lemma we then have equality in (2.2),

$$\left|\frac{dw}{dz}\right| = \frac{1-|w|^2}{1-|z|^2}.$$

In differential form this becomes

$$\frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2},$$