with equality only when $f(z)$ is a multiple of $z - z_0$. This can be proved directly, based on the factorization $f(z) = (z - z_0)g(z)$. It can also be obtained from (1.1) by scaling in both the z-variable and the w-variable, $w = f(z)$, and by translating the center of the disk to z_0, as follows. The change of variable $\zeta \mapsto R\zeta + z_0$ maps the unit disk $\{|\zeta| < 1\}$ onto the disk $\{|z - z_0| < R\}$. If we define $h(\zeta) = f(R\zeta + z_0)/M$, then $h(\zeta)$ is analytic on the open unit disk and satisfies $|h(\zeta)| \leq 1$ and $h(0) = 0$. The estimate $|h(\zeta)| \leq |\zeta|$ becomes (1.2).

The Schwarz lemma gives an explicit estimate for the "modulus of continuity" of an analytic function. It shows that a uniformly bounded family of analytic functions is "equicontinuous" at each point. We will return in Chapter XI to treat the ideas of equicontinuity and compactness for families of analytic functions.

There is an infinitesimal version of the Schwarz lemma.

Theorem. *Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \leq 1$ for $|z| < 1$, and $f(0) = 0$, then*

$$|f'(0)| \leq 1, \tag{1.3}$$

with equality if and only if $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$.

The estimate (1.3) follows by taking $z \to 0$ in the Schwarz lemma. For the case of equality, we consider the factorization $f(z) = zg(z)$ used in the proof of the Schwarz lemma, and we observe that $g(0) = f'(0)$. If $|f'(0)| = 1$, we then have $|g(0)| = 1$, and we conclude as before from the strict maximum principle that $g(z)$ is constant. Hence $f(z) = \lambda z$.

Note that the estimate (1.3) is the same as the Cauchy estimate for $f'(0)$ derived in Section IV.4, without the hypothesis that $f(0) = 0$. See also Exercise 7.

Exercises for IX.1

1. Let $f(z)$ be analytic and satisfy $|f(z)| \leq M$ for $|z - z_0| < R$. Show that if $f(z)$ has a zero of order m at z_0, then
$$|f(z)| \leq \frac{M}{R^m}|z - z_0|^m, \qquad |z - z_0| < R.$$
Show that equality holds at some point $z \neq z_0$ only when $f(z)$ is a constant multiple of $(z - z_0)^m$.

2. Suppose that $f(z)$ is analytic and satisfies $|f(z)| \leq 1$ for $|z| < 1$. Show that if $f(z)$ has a zero of order m at z_0, then $|z_0|^m \geq |f(0)|$. *Hint.* Let $\psi(z) = (z - z_0)/(1 - \overline{z_0}z)$, which is a fractional linear transformation mapping the unit disk onto itself, and show that $|f(z)| \leq |\psi(z)|^m$.

3. Suppose that $f(z)$ is analytic for $|z| \leq 1$, and suppose that $1 < |f(z)| < M$ for $|z| = 1$, while $f(0) = 1$. Show that $f(z)$ has a zero in the unit disk, and that any such zero z_0 satisfies $|z_0| > 1/M$. *Hint.* For the second assertion, consider $\psi(f(z))$, where $\psi(w)$ is a fractional linear transformation mapping 1 to 0 and the circle $\{|w| = M\}$ to the unit circle. Or use Exercise 2.

4. Suppose that $f(z)$ is analytic for $|z| < 1$ and satisfies $f(0) = 0$ and $\operatorname{Re} f(z) < 1$. (a) Show that $|f(z)| \leq 2|z|/(1 - |z|)$. *Hint.* Consider the composition of $f(z)$ and the fractional linear transformation mapping the half-plane $\{\operatorname{Re} w < 1\}$ onto the unit disk. (b) Show that $|f'(0)| \leq 2$. (c) For fixed z_0 with $0 < |z_0| < 1$, determine for which functions $f(z)$ there is equality in (a). (d) Determine for which functions $f(z)$ there is equality in (b). (e) By scaling the estimates in (a) and (b), obtain sharp estimates for $|g(z)|$ and $|g'(0)|$ for functions $g(z)$ analytic for $|z| < R$ and satisfying $g(0) = 0$ and $\operatorname{Re} g(z) < C$.

5. Suppose that $f(z)$ is analytic and satisfies $|f(z)| \leq 1$ for $|z| < 1$. Show that if $|f(0)| \geq r$, then $|f(z)| \geq (r - |z|)/(1 - r|z|)$ for $|z| < r$. Determine for which functions $f(z)$ equality holds at some point z_0 with $|z_0| < r$.

6. Let $f(z)$ be a conformal map of the open unit disk onto a domain D. Show that the distance from $f(0)$ to the boundary of D is estimated by $\operatorname{dist}(f(0), \partial D) \leq |f'(0)|$.

7. Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is analytic for $|z| < 1$ and satisfies $|f(z)| \leq M$.
   (a) Show that $\sum_{k=0}^{\infty} |a_k|^2 \leq M^2$. *Hint.* Integrate $|f(z)|^2$ around a circle of radius r.
   (b) Show using (a) that $|f'(0)| \leq M$, with equality only if $f(z)$ is a constant multiple of z. *Remark.* It is not assumed that $f(0) = 0$.
   (c) Show that $|f^{(k)}(0)| \leq k!M$, with equality only if $f(z)$ is a constant multiple of z^k.

8. Suppose that $f(z)$ is analytic for $|z| < 1$ and satisfies $|f(z)| < 1$, $f(0) = 0$, and $|f'(0)| < 1$. Let $r < 1$. Show that there is a constant $c < 1$ such that $|f(z)| \leq c|z|$ for $|z| \leq r$. Show that the nth iterate $f_n(z) = f(f(\cdots f(z) \cdots)) = f(f_{n-1}(z))$ of $f(z)$ satisfies $|f_n(z)| \leq c^n|z|$ for $|z| \leq r$. Deduce that $f_n(z)$ converges to zero normally on the open unit disk $\mathbb{D}$.