the series. Since

$$\int_0^z \frac{d\zeta}{(\zeta - k)^2} = -\left(\frac{1}{z-k} + \frac{1}{k}\right), \qquad k \neq 0,$$

$$\int_0^z \left(\frac{\pi^2}{\sin^2(\pi z)} - \frac{1}{\zeta^2}\right) d\zeta = -\pi\cot(\pi z) + \frac{1}{z},$$

we obtain from (2.1) the partial fractions decomposition

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{k\neq 0}\left(\frac{1}{z-k} + \frac{1}{k}\right).$$

If we combine the terms for $\pm k$, the constants cancel, and we obtain

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{z-k} + \frac{1}{z+k}\right) = \frac{1}{z} + 2z\sum_{k=1}^{\infty}\frac{1}{z^2 - k^2}.$$

Exercises for XIII.2

1. Use the partial fractions decomposition of $\pi^2/\sin^2(\pi z)$ to establish the formula

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. Establish the partial fractions decomposition

$$\pi\tan(\pi z) = -2z\sum_{n=0}^{\infty}\frac{1}{z^2 - \left(n+\frac{1}{2}\right)^2}.$$

Hint. Use $\tan w = \cot w - 2\cot(2w)$.

3. Show that

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-1)^n}{z^2 - n^2}.$$

4. Show that

$$\frac{\pi}{\cos(\pi z)} = \sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)}{z^2 - \left(n-\frac{1}{2}\right)^2}.$$

5. Let $\{z_k\}$ be a sequence of distinct points such that $|z_k| \to \infty$ and $\sum |z_k|^{-m-1} < \infty$. Show that $z^m \sum 1/z_k^m(z-z_k)$ converges normally to a meromorphic function with principal part $1/(z-z_k)$ at z_k. (If $z_k = 0$, we replace the corresponding summand by $1/z$.)

6. Construct a meromorphic function on the complex plane whose poles are simple poles at the Gaussian integers $m+ni$ with residue 1.

7. Construct a meromorphic function on the complex plane whose poles are double poles at the points $\log n$, $n \geq 1$, with principal parts $1/(z - \log n)^2 + 1/(z - \log n)$.

8. Construct a meromorphic function on the open unit disk $\mathbb{D}$ whose poles are simple poles at the points $(1 - 2^{-n})e^{2\pi i k/n}$, $1 \leq k \leq n$, $n \geq 1$, with residue 1.

9. Show that $\sum_{-\infty}^{\infty} 1/(z^3 - n^3)$ converges normally to a meromorphic function. Locate the poles and find the corresponding principal parts of the function. Express the function in terms of trigonometric functions (specifically, the cotangent function).

10. Show that the lattice points $m\omega_1 + n\omega_2$, $-\infty < m, n < \infty$, can be arranged in a sequence $\{z_k\}_{k=0}^{\infty}$ such that $|z_k| \geq c\sqrt{k}$.

11. Let $\{z_k\}$ be a sequence of distinct points such that $|z_k| > c\sqrt{k}$. Show that
$$\sum \left[\frac{1}{(z - z_k)^2} - \frac{1}{z_k^2} \right]$$
converges normally on $\mathbb{C}$ and absolutely at each $z \in \mathbb{C}$.

12. Let $f(z)$ be a doubly periodic meromorphic function on $\mathbb{C}$ with periods ω_1 and ω_2, and let $\mathcal{P}(z)$ be the Weierstrass P-function associated with the periods ω_1 and ω_2. (a) Show that if the only poles of $f(z)$ are double poles at the lattice points $m\omega_1 + n\omega_2$, $-\infty < m, n < \infty$, then there are constants a and b such that $f(z) = a\mathcal{P}(z) + b$. (b) Show that if the only poles of $f(z)$ are triple poles at the lattice points $m\omega_1 + n\omega_2$, $-\infty < m, n < \infty$, then there are constants a, b, c such that $f(z) = a\mathcal{P}(z) + b\mathcal{P}'(z) + c$. (c) Show that $\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 + a\mathcal{P}(z)^2 + b\mathcal{P}(z) + c$ for some constants a, b, c.

13. Let D be a domain in $\mathbb{C}$, and let $E_k = \{|z - z_k| \leq r_k\}$, $k \geq 1$, be disjoint closed disks in D that accumulate only on the boundary of D. Suppose $Q_k(z)$ is analytic for $|z - z_k| > r_k$. Show that there is an analytic function $f(z)$ on $D \backslash \cup_{k=1}^{\infty} E_k$ such that for each k, $f(z) - Q_k(z)$ extends analytically to E_k.

3. Infinite Products

An **infinite product** is an expression of the form $\prod_{j=1}^{\infty} p_j$, where the p_j's are complex numbers. We say that the infinite product **converges** if $p_j \to 1$ and $\sum \operatorname{Log} p_j$ converges, where we sum only over terms for which $p_j \neq 0$. If the infinite product converges, we define its value to be 0 if one