$f(z_0) = 0$. The family $\mathcal{F}$ is nonempty, since if $h(z)$ maps D conformally onto a bounded domain, then the function $f(z) = \varepsilon(h(z) - h(z_0))$ is in $\mathcal{F}$ for $\varepsilon > 0$ small. We consider the extremal problem of maximizing $|f'(z_0)|$ over $f \in \mathcal{F}$. As before, set

$$A = \sup\{|f'(z_0)| : f \in \mathcal{F}\} > 0,$$

and let $\{f_n(z)\}$ be a sequence of functions in $\mathcal{F}$ such that $|f_n'(z_0)| \to A$. By Montel's theorem, the f_n's have a subsequence that converges normally on D to an analytic function $\varphi(z)$. Clearly, $|\varphi(z)| \leq 1$, and $|\varphi'(z_0)| = A$. The functions f_n are univalent, so by Hurwitz's theorem (Section VIII.2), either φ is constant or φ is univalent. Since $\varphi'(z_0) \neq 0$, φ is nonconstant, and consequently, φ is univalent, mapping D onto a subdomain of $\mathbb{D}$. We claim that $\varphi(D) = \mathbb{D}$. Otherwise, we could apply the preceding Lemma to the domain $\varphi(D)$ and find a conformal map $\psi(\zeta)$ of $\varphi(D)$ onto a subdomain of $\mathbb{D}$ such that $\psi(0) = 0$ and $|\psi'(0)| > 1$. Then $\psi \circ \varphi \in \mathcal{F}$ would satisfy $|(\psi \circ \varphi)'(z_0)| = |\psi'(0)\varphi'(z_0| > A$, contradicting the definition of A. This completes the proof.

We will give another proof of the Riemann mapping theorem in Section XV.5, based on the solvability of the Dirichlet problem.

Exercises for XI.6

1. Find explicitly the functions $f(\zeta)$ and $g(\zeta)$ used in the proof of the lemma. Show by computing the derivative that $|(f \circ h \circ g)'(0)| > 1$.

2. Let $\varphi(z)$ be the Riemann map of a simply connected domain D onto the open unit disk, normalized by $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$. Show that if $f(z)$ is any analytic function on D such that $|f(z)| \leq 1$ for $z \in D$, then $|f'(z_0)| \leq \varphi'(z_0)$, with equality only when $f(z)$ is a constant multiple of $\varphi(z)$. *Remark.* This shows that $\varphi(z)$ is the Ahlfors function of D corresponding to z_0.

3. Let $\varphi(z)$ be the Riemann map of a simply connected domain D onto the open unit disk, normalized by $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$. Show that if $f(z)$ is any analytic function on D such that $|f(z)| \leq 1$ for $z \in D$, then $\operatorname{Re} f'(z_0) \leq \varphi'(z_0)$, with equality only when $f(z) = \varphi(z)$.

4. Let $\{D_m\}$ be an increasing sequence of simply connected domains, and let φ_m be the Riemann map of D_m onto the open unit disk $\mathbb{D}$, normalized so that $\varphi_m(z_0) = 0$ and $\varphi_m'(z_0) > 0$ for some fixed $z_0 \in D_1$. Let D be the union of the D_m's. Show that if D is the entire complex plane, then the φ_m's are eventually defined on any disk $\{|z| \leq R\}$ and converge there uniformly to 0. Otherwise, D is simply connected and the φ_m's are eventually defined on each compact subset of D and converge there uniformly to the Riemann map φ of D onto $\mathbb{D}$ satisfying $\varphi(z_0) = 0$ and $\varphi'(z_0) > 0$.

5. Let $\{D_m\}$ be a decreasing sequence of simply connected domains, and suppose $w_0 \in D_m$ for all m. Let $g_m(z)$ be the conformal map of the open unit disk $\mathbb{D}$ onto D_m, normalized so that $g_m(0) = w_0$ and $g_m'(0) > 0$. Show that $g_m(z)$ converges normally on $\mathbb{D}$ to a function $g(z)$. If the distance from w_0 to the boundary of D_m tends to 0, then $g(z)$ is the constant function w_0, and otherwise, $g(z)$ maps $\mathbb{D}$ conformally onto some simply connected domain D. Describe D in terms of the D_m's.

6. Show that the function $\zeta \mapsto \zeta^2$ is not a strict contraction of the hyperbolic disk, that is, show that there is no constant $c < 1$ such that $\rho(\zeta^2, \xi^2) \leq c\rho(\zeta, \xi)$ for all $\zeta, \xi \in \mathbb{D}$. *Remark.* See Exercise IX.3.9.

7. Suppose $f : \mathbb{D} \to \mathbb{D}$ is an analytic function from the unit disk into itself with a fixed point at $z_0 \in \mathbb{D}$. Show that the stretching at z_0 of $f(z)$ in the hyperbolic metric is the same as the stretching at z_0 of $f(z)$ in the Euclidean metric,

$$\lim_{z \to z_0} \frac{\rho(f(z), z_0)}{\rho(z, z_0)} = \lim_{z \to z_0} \frac{|f(z) - z_0|}{|z - z_0|} = |f'(z_0)|.$$