1. Find the poles and residues of the following functions

\[
\frac{1}{z^4 + 5z^2 + 6}, \quad \frac{1}{(z^2 - 1)^2}, \quad \frac{\pi \cot(\pi z)}{z^2}, \quad \frac{1}{z^m(1 - z)^n} \quad (m, n \in \mathbb{Z}_{>0})
\]

\textit{Solution:} Throughout we use the following formula for calculating residues: If \( f(z) \) has a pole of order \( k \) at \( z = z_0 \) then

\[
\text{res}(f, z_0) = \frac{1}{(k - 1)!} \left. \frac{d^{k-1}}{dz^{k-1}} \left( (z - z_0)^k f(z) \right) \right|_{z = z_0}.
\]

In particular, if \( f(z) \) has a simple pole at \( z_0 \) then the residue is given by simply evaluating the non-polar part: \( (z - z_0)f(z) \), at \( z = z_0 \) (or by taking a limit if we have an indeterminate form).

Let

\[
f(z) := \frac{1}{z^4 + 5z^2 + 6} = \frac{1}{(z^2 + 2)(z^2 + 3)} = \frac{1}{(z + i\sqrt{2})(z - i\sqrt{2})(z + i\sqrt{3})(z - i\sqrt{3})}.
\]

This has simple poles at \( z = \pm i\sqrt{2}, \pm i\sqrt{3} \) with residues

\[
\text{res}(f, \pm i\sqrt{2}) = \left. \frac{1}{z^4 + 5z^2 + 6} \right|_{z = \pm i\sqrt{2}} = \left. \frac{1}{(z + i\sqrt{2})(z^2 + 3)} \right|_{z = \pm i\sqrt{2}} = \frac{1}{2i\sqrt{2}},
\]

\[
\text{res}(f, \pm i\sqrt{3}) = \frac{1}{2i\sqrt{3}},
\]

\[
\text{res}(f, \pm i\sqrt{3}) = \frac{1}{2i\sqrt{3}}.
\]
For the second one let
\[ f(z) = \frac{1}{(z^2 - 1)^2} = \frac{1}{(z+1)^2(z-1)^2}. \]
This has double poles at $\pm 1$. From the formula we get
\[
\text{res}(f, 1) = \left. \frac{d}{dz} \frac{1}{(z+1)^2} \right|_{z=1} = -1/4,
\]
\[
\text{res}(f, -1) = \left. \frac{d}{dz} \frac{1}{(z-1)^2} \right|_{z=-1} = 1/4.
\]

For the third let
\[ f(z) = \frac{\pi \cot(\pi z)}{z^2}. \]
Now, $\cot(\pi z)$ has poles wherever $\sin(\pi z) = 0$, so at $z = n \in \mathbb{Z}$. About these points we have
\[
\sin(\pi z) = \sin(\pi n) + \pi \cos(\pi n)(z - n) - \pi^2 \sin(\pi n) \frac{(z - n)^2}{2!} - \pi^3 \cos(\pi n) \frac{(z - n)^3}{3!} + \cdots
\]
\[
= (-1)^n \pi (z - n) \left[ 1 - \pi^2 \frac{(z - n)^2}{3!} + O((z - n)^4) \right]
\]
and
\[
\cos(\pi z) = \cos(\pi n) - \pi \sin(\pi n)(z - n) - \pi^2 \cos(\pi n) \frac{(z - n)^2}{2!} + \pi^3 \sin(\pi n) \frac{(z - n)^3}{3!} + \cdots
\]
\[
= (-1)^n \left[ 1 - \pi^2 \frac{(z - n)^2}{2!} + O((z - n)^4) \right].
\]
Hence, for $z$ close to $n \in \mathbb{Z}$, we have
\[
\cot(\pi z) = \frac{1 - \pi^2 (z - n)^2/2 + O((z - n)^4)}{\pi(z - n) \left[ 1 - \pi^2 (z - n)^2/6 + O((z - n)^4) \right]}
\]
\[
= \frac{1 - \pi^2 (z - n)^2/2 + O((z - n)^4)}{\pi(z - n)} \left[ 1 + \pi^2 (z - n)^2/6 + O((z - n)^4) \right]
\]
\[
= \frac{1}{\pi(z - n)} - \frac{\pi}{3} (z - n) + O((z - n)^3)
\]
Therefore, $f(z) = \pi \cot(\pi z)/z^2$ has simple poles at $z = n \neq 0$ and a triple pole at $z = 0$. For the simple poles we have
\[
\text{res}(f, n) = (z - n) \left. \frac{\pi \cot(\pi z)}{z^2} \right|_{z=n} = \frac{1}{n^2}.
\]
For the triple pole at $z = 0$ we have

$$f(z) = \frac{1}{z^3} - \frac{\pi^2}{3} \frac{1}{z} + O(z)$$

so the residue is $-\pi^2/3$.

Finally, the function

$$f(z) = \frac{1}{z^m(1 - z)^n}$$

has a pole of order $m$ at $z = 0$ and a pole of order $n$ at $z = 1$. From the formula for residues we have

$$\text{res}(f, 0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left( \frac{1}{1-z^n} \right) \bigg|_{z=0} = \frac{n(n+1) \cdots (n+m-2)}{(m-1)!}$$

and

$$\text{res}(f, 1) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left( \frac{(-1)^n}{z^m} \right) \bigg|_{z=1} = \frac{-m(m+1) \cdots (m+n-2)}{(n-1)!} = -\frac{(m+n-2)!}{(m-1)!(n-1)!} = -\text{res}(f, 0).$$

2. Use the substitution $e^{i\theta} = z$ along with the residue theorem to show that

$$\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{2\pi}{\sqrt{3}}.$$ 

**Solution:** As suggested we let $e^{i\theta} = z$ so that $d\theta = dz/(iz)$ and the integral becomes

$$\frac{1}{i} \int_{|z|=1} \frac{dz}{z(2 + (z + z^{-1})/2)} = \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2 + 4z + 1}.$$ 

Now $z^2 + 4z + 1$ has zeros of order 1 at $z = z_{\pm} = -2 \pm \sqrt{3}$ and so the integrand has simple poles at $z_+$ and $z_-$. Only $z_+$ lies in the unit disk and therefore by the residue
Theorem
\[ \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2 + 4z + 1} = \frac{2}{i} \times 2\pi i \times \text{res} \left( \frac{1}{z^2 + 4z + 1}, z_+ \right) \]
\[ = 4\pi (z - z_+) \left. \frac{1}{z^2 + 4z + 1} \right|_{z=z_+} \]
\[ = 4\pi (z - z_+) \left. \frac{1}{(z - z_+)(z - z_-)} \right|_{z=z_+} \]
\[ = 4\pi \frac{1}{z_+ - z_-} \]
\[ = 4\pi \frac{1}{2\sqrt{3}} \]
\[ = \frac{2\pi}{\sqrt{3}}. \]

3. Evaluate the following integrals via residues. Show all estimates.
(i) \[
\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx
\]
(ii) \[
\int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx, \ a \text{ real}
\]
(iii) \[
\int_0^\infty \frac{\log x}{1 + x^2} \, dx
\]

Solution: (i) Since the integrand is an even function the integral in question is equal to \( I/2 \) where
\[ I = \int_{-\infty}^\infty \frac{x^2}{x^4 + 5x^2 + 6} \, dx. \]
As a function of a complex variable, the integrand has simple poles at \( \pm i\sqrt{2}, \pm i\sqrt{3} \). We will be considering a semicircular contour in the upper half plane so we only need calculate the residues at \( z = i\sqrt{2}, i\sqrt{3} \). A slight modification of the first calculation in question 1 gives
\[ \text{res} \left( \frac{z^2}{z^4 + 5z^2 + 6}, i\sqrt{2} \right) = \frac{(i\sqrt{2})^2}{2i\sqrt{2}} = \frac{i\sqrt{2}}{2} \]
and
\[
\text{res}\left(\frac{z^2}{z^4 + 5z^2 + 6}, i\sqrt{3}\right) = -\frac{(i\sqrt{3})^2}{2i\sqrt{3}} = -\frac{i\sqrt{3}}{2}.
\]

Now, consider the semicircular contour \(\Gamma_R\), which starts at \(R\), traces a semicircle in the upper half plane to \(-R\) and then travels back to \(R\) along the real axis. Then, on taking \(R\) large enough, by the residue theorem
\[
\int_{\Gamma_R} \frac{z^2}{z^4 + 5z^2 + 6} \, dz = 2\pi i \times \sum \text{residues inside } \Gamma_R = 2\pi i\left(\frac{i\sqrt{2}}{2} - \frac{i\sqrt{3}}{2}\right) = \pi(\sqrt{3} - \sqrt{2}).
\]
On the other hand
\[
\lim_{R \to \infty} \int_{\Gamma_R} \frac{z^2}{z^4 + 5z^2 + 6} \, dz = I + \lim_{R \to \infty} \int_{\gamma_R} \frac{z^2}{z^4 + 5z^2 + 6} \, dz
\]
where \(\gamma_R\) is the semicircle in the upper half plane. But by the Estimation Lemma
\[
\left|\int_{\gamma_R} \frac{z^2}{z^4 + 5z^2 + 6} \, dz\right| \leq \pi R \max_{z \in \gamma_R} \left|\frac{z^2}{z^4 + 5z^2 + 6}\right| \ll \frac{1}{R} \to 0
\]
as \(R \to \infty\). Hence, \(I = \pi(\sqrt{3} - \sqrt{2})\) and so
\[
\int_{0}^{\infty} \frac{x^2}{x^4 + 5x^2 + 6} \, dx = \frac{\pi}{2}(\sqrt{3} - \sqrt{2}).
\]
The last estimate in the inequality for the integral over \(\gamma_R\) should be clear since the dominant term is the \(z^4\) term in the denominator, but for completeness:
\[
\pi R \max_{z \in \gamma_R} \left|\frac{z^2}{z^4 + 5z^2 + 6}\right| = \pi \max_{z \in \gamma_R} \left|\frac{1}{1 + 5z^{-2} + 6z^{-4}}\right|
\ll \frac{\pi}{R} \max_{z \in \gamma_R} \frac{1}{1 - 5|z|^{-2} - 6|z|^{-4}}
\ll \frac{\pi}{R} \frac{1}{1 - 5R^{-2} - 6R^{-4}}
\]
where we have used \(|z + w| \geq |z| - |w|\) in the second line. On taking \(R \geq 5\), say, this is \(\ll 2\pi/R\) and the constant \(2\pi\) is absorbed into the \(\ll\) symbol.

(ii). We have
\[
\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} \, dx = \frac{1}{2} \Im \left(\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} \, dx\right).
\]
Denote this last integral by \(J\). Again, we will consider \(J\) as the horizontal section of the contour \(\Gamma_R\) from part (i).
In the upper half plane the integrand has a simple pole at \( z = ia \) with residue
\[
\text{res}\left( \frac{ze^{iz}}{z^2 + a^2}, ia \right) = (z - ia) \left. \frac{ze^{iz}}{z^2 + a^2} \right|_{z=ia} = \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}.
\]

Hence, by the residue theorem
\[
\pi ie^{-a} = \lim_{R \to \infty} \int_{\Gamma_R} \frac{ze^{iz}}{z^2 + a^2} \, dz = J + \lim_{R \to \infty} \int_{\gamma_R} \frac{ze^{iz}}{z^2 + a^2} \, dz.
\]
Thus it remains to show that this last integral vanishes in the limit. This is similar to question 7 (ii) of Problems 3; a trivial estimate of the integrand is \( \ll 1/R \) which is not enough for the Estimation Lemma. Instead we apply integration by parts which is probably the quickest way (see Problems 3). Integrating \( e^{iz} \) and differentiating the rest gives
\[
\int_{\gamma_R} \frac{ze^{iz}}{z^2 + a^2} \, dz = \left. \frac{ze^{iz}}{i(z^2 + a^2)} \right|_{z=R}^{z=-R} - \frac{1}{i} \int_{\gamma_R} \left( \frac{1}{z^2 + a^2} - \frac{2z^2}{(z^2 + a^2)^2} \right) e^{iz} \, dz.
\]
The first term on the right is \( -2R \cos R/i(R^2 + a^2) \ll 1/R \). For the integrals we use the Estimation Lemma to give
\[
\left| \int_{\gamma_R} \frac{e^{iz}}{z^2 + a^2} \, dz \right| \leq \pi R \max_{z \in \gamma_R} \left| \frac{e^{iz}}{z^2 + a^2} \right| \leq \pi R \frac{1}{R^2 - a^2} \ll \frac{1}{R},
\]
\[
\left| \int_{\gamma_R} \frac{z^2 e^{iz}}{(z^2 + a^2)^2} \, dz \right| \leq \pi R \max_{z \in \gamma_R} \left| \frac{z^2 e^{iz}}{(z^2 + a^2)^2} \right| \leq \pi R^3 \frac{1}{(R^2 - a^2)^2} \ll \frac{1}{R},
\]
as \( R \to \infty \). Hence, \( J = \pi ie^{-a} \) and so
\[
\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} \, dx = \frac{1}{2} \Im(J) = \frac{\pi}{2} e^{-a}.
\]

(iii) This is quite hard and, as I discovered recently, the solution is in Conway’s book anyway (pg. 117–118). My bad. It’s still instructive to attempt this before reading Conway though.

We see that, as a function of a complex variable, the integrand has a branch cut and simple poles at \( z = i, -i \). Taking the branch of the log with \(-\pi < \arg(z) < \pi\), we would like to choose a contour which lies just above and below the cut and that also picks up the residues at \( i, -i \). A natural choice is the contour we used in the lectures. This consisted of a small circle about \( z = 0 \), horizontal lines just above and below the negative real axis, and a large circle completing the contour.
Unfortunately, with this choice the integrals over the horizontal lines $L_1, L_2$ are given by

\[
\begin{align*}
\int_{L_1} \log z \, dz + \int_{L_2} \log z \, dz &= \int_{-\infty}^0 \frac{\log(x + i\delta)}{1 + (x + i\delta)^2} \, dx + \int_0^{-\infty} \frac{\log(x - i\delta)}{1 + (x - i\delta)^2} \, dx \\
&\quad + \int_{-\infty}^0 \frac{\log |x| + \pi i}{1 + x^2} \, dx + \int_0^{-\infty} \frac{\log |x| - \pi i}{1 + x^2} \, dx \\
&= 2\pi i \int_{-\infty}^0 \frac{1}{1 + x^2} \, dx
\end{align*}
\]

and the integral we’re after has disappeared.

This motivates the choice of a new contour: we want something with a horizontal line over the whole real axis, since then the integral over this line is given by

\[
\int_{-\infty}^\infty \log |x| \, dx = 2 \int_0^\infty \frac{\log x}{1 + x^2} \, dx
\]

and we avoid the problem of cancellation. Consequently, we choose a branch of $\log z$ with a branch cut along the negative imaginary axis. To avoid $z = 0$ and the branch cut we indent our contour with a small circle in the upper half plane. We complete the contour with a large circle.

Thus, let $\Gamma_{r,R}$ be the contour consisting of a line from $r$ to $R$, a semicircle in the upper half plane traced from $R$ to $-R$, a line from $-R$ to $-r$, and a semicircle in the upper half plane traced from $-r$ to $r$. Following the details in Conway we find

\[
\int_{\Gamma_{r,R}} \frac{\pi \cot(\pi z)}{z^2} \, dz = 0.
\]

4. Let $\Gamma_N$ be the square which crosses the real axis at $\pm (N + 1/2)$ with $N \in \mathbb{N}$.

(i) Show that $\cot(\pi z)$ is bounded on $\Gamma_N$ and hence show that

\[
\lim_{N \to \infty} \int_{\Gamma_N} \frac{\pi \cot(\pi z)}{z^2} \, dz = 0.
\]

(ii) For a given $N$ compute the above integral via residues. Conclude something interesting.

Solution: (i) We have

\[
\cot(\pi z) = i \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} = i \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} = i \frac{e^{2\pi ix} e^{-2\pi y} + 1}{e^{2\pi ix} e^{-2\pi y} - 1}.
\]
Thus, if \( y \geq 1 \)

\[
|\cot(\pi z)| \leq \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \leq \frac{1 + e^{-2\pi}}{1 - e^{-2\pi}}.
\]

Since \( \cot(\pi z) = \cot(\pi z) \) we obtain the same bound for \( y \leq -1 \). It remains to show that \( \cot(\pi z) \) is bounded for \( z = \pm(N + 1/2) + iy \) with \( |y| < 1 \). But in this case

\[
|\cot(\pi z)| = \left| \frac{e^{\pm 2\pi i (N + 1/2)} e^{-2\pi y} + 1}{e^{\pm 2\pi i (N + 1/2)} e^{-2\pi y} - 1} \right| = \left| \frac{-e^{-2\pi y} + 1}{-e^{-2\pi y} - 1} \right| < \frac{e^{2\pi} - 1}{e^{-2\pi} + 1} =: C.
\]

Since \( C \) is bigger than the bound for \( |y| \geq 1 \), we may say that \( |\cot(\pi z)| < C \) on all of \( \Gamma_N \).

(ii) By the estimation lemma

\[
\left| \int_{\Gamma_N} \frac{\pi \cot(\pi z)}{z^2} \, dz \right| \leq 8\pi(N + 1/2) \max_{z \in \Gamma_N} \left| \frac{\cot(\pi z)}{z^2} \right| < 8\pi(N + 1/2) \frac{C}{(N + 1/2)^2} = \frac{8\pi C}{N + 1/2}
\]

and this tends to zero as \( N \to \infty \).

On the other hand, for a fixed \( N \) this integral is given by \( 2\pi i \times \) the sum of residues inside \( \Gamma_N \). Using the results from question 1 then gives

\[
\int_{\Gamma_N} \frac{\pi \cot(\pi z)}{z^2} \, dz = 2\pi i \left( \sum_{-N \leq n \leq N, n \neq 0} \frac{1}{n^2} - \frac{\pi^2}{3} \right).
\]

Therefore, on letting \( N \to \infty \) we have

\[
0 = \sum_{-\infty < n < \infty, n \neq 0} \frac{1}{n^2} - \frac{\pi^2}{3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{3}
\]

and so

\[
\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},
\]

which is interesting.

5. (i) Let \( f(z) = z^6 + \cos z \). Find the change in argument of \( f(z) \) as \( z \) travels once around the circle of radius 2, center zero, in the positive direction.

(ii) How many solutions does \( 3e^z - z = 0 \) have in the disk \( |z| \leq 1? \)

(iii) Use Rouche’s Theorem to prove that a polynomial of degree \( n \) has \( n \) roots in \( \mathbb{C} \).
**Solution:** (i) By the argument principle, the change in argument of $f(z)$ as $z$ travels around the circle is equal to $(2\pi \times \text{number of zeros minus poles of } f(z) \text{ inside the circle})$, so just the number of zeros then.

To find the number of zeros of $f(z)$ inside the circle we compare $f(z)$ with its dominant term $z^6$ and apply Rouche’s Theorem. So let $g(z) = z^6$. On the circle we have

$$|g(z)| = |z|^6 = 2^6 = 64$$

and

$$|f(z) - g(z)| = |\cos(z)| \leq \frac{|e^{iz}| + |e^{-iz}|}{2} \leq e|z| = e^2 < 64.$$ 

Thus, Rouche’s Theorem applies and $f(z)$ has the same number of zeros inside the circle as $z^6$, which is 6. Hence, the change in argument is $2\pi \times 6 = 12\pi$.

(ii). Rephrasing the question we ask for the number of zeros of $f(z) = 3e^z - z$ in the closed unit disk. We apply Rouche’s Theorem again. This time the dominant term is $g(z) = 3e^z$. On the unit circle we have

$$|g(z)| = 3e^{\Re(z)} \geq 3e^{-1} > 1$$

and

$$|f(z) - g(z)| = |z| = 1.$$ 

By Rouche’s Theorem $3e^z - z$ has the same number of zeros in the unit disk as $3e^z$, which is none. So the answer is no solutions.

(iii) In the lectures we showed that the polynomial $z^6 + z + 2$ has 6 zeros in the disk $|z| \leq 2$ by comparing it with $z^6$ and applying Rouche’s Theorem. We generalise this by comparing a general polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0$$

with its leading term

$$g(z) = a_n z^n$$

on arbitrarily large circles. On the circle $|z| = R > 1$ we have

$$|g(z)| = |a_n|R^n$$
and
\[ |f(z) - g(z)| = |a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0| \]
\[ \leq |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} + \cdots + |a_0| \]
\[ < n \max_{0 \leq j \leq n-1} |a_j|R^{n-1} \]
\[ = \left( \frac{n \max_{j=0}^{n-1} |a_j|}{|a_n|} R \right) |a_n|R^n \]

Therefore, on a circle of radius \( R \geq n \max_{j=0}^{n-1} |a_j|/|a_n| \) we have \(|f(z) - g(z)| < |g(z)|\). By Rouche’s Theorem \( f(z) \) has the same number of zeros inside the circle as \( g(z) \), which is \( n \).

6. Prove that the function
\[ f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(n + z)^2} \]
is meromorphic on \( \mathbb{C} \).

**Solution:** Clearly, \( f(z) \) has poles only at the integers so we need to show that \( f(z) \) is analytic on \( \mathbb{C}\setminus\mathbb{Z} \). By the results given in the lectures, this will follow if we can show that the series is uniformly convergent on compact subsets of said region. Since
\[ f(z) = \frac{1}{z^2} + \sum_{n \neq 0} \frac{1}{(n + z)^2} \]
it suffices to show the uniform convergence of this last sum.

So let \( |z| \leq R \), let \( z \) be fixed away from integers: \(|z + n| \geq \rho > 0 \ \forall n \in \mathbb{Z} \), and let \(-M\) be the nearest integer to \( z \). Note that the series looks like \( 2 \sum_{n \neq 0} n^{-2} \) which is convergent. So we first pull out the convergent part by writing
\[ \sum_{n \neq 0} \frac{1}{(n + z)^2} = \sum_{n \neq 0} \frac{1}{n^2(1 + z/n)^2}. \]

Then if \(|n| \leq |M|\)
\[ |z + n| \geq |z + m| \implies |1 + z/n| \geq (|M|/|n|)|1 + z/m| \geq |1 + z/M| \geq \rho/|M|. \]

If \(|n| > |M| + 1\)
\[ |1 + z/n| \geq |1 + \Re(z)/n| \geq 1 - \frac{\left|\Re(z)\right|}{|n|} > 1 - \frac{|M| + 1}{|M| + 2} = \frac{1}{|M| + 2}. \]
Therefore,
\[
\sum_{n \neq 0} \left| \frac{1}{(n + z)^2} \right| \leq \frac{|M|^2}{\rho^2} \sum_{0 \neq |n| \leq |M|} \frac{1}{n^2} + (|M| + 2)^2 \sum_{|n| > |M| + 1} \frac{1}{n^2} + \frac{1}{|z \pm (M \pm 1)|^2} \\
\leq 2\zeta(2)\frac{(R + 1)^2}{\rho^2} + 2\zeta(2)(R + 3)^2 + \frac{4}{\rho^2} < \infty.
\]

Hence, by the Weierstrass $M$ test with $M_n = n^{-2} \max((R + 3)^2, (R + 1)^2/\rho^2)$ the series is uniformly convergent on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

7. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function whose derivatives $\varphi^{(k)}(t)$, $k \geq 0$, are of rapid decay at $\infty$ i.e.

\[
\lim_{t \to \infty} t^A \varphi^{(k)}(t) = 0
\]

for all $A \in \mathbb{R}$ and all $k \geq 0$. The Mellin transform of $\varphi$ is defined as

\[
\tilde{\varphi}(z) = \int_0^\infty \varphi(t)t^{z-1}dt.
\]

(i) Prove that $\tilde{\varphi}(z)$ is analytic in the region $\Re(z) > 0$.

(ii) Use integration by parts to show that $\tilde{\varphi}(z)$ can be analytically continued to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, with possible simple poles at $z = -n$.

(iii) Find the residues at these poles and compute the formal sum

\[
\sum_{n=0}^{\infty} \text{res}(\tilde{\varphi}(z)t^{-z}, -n).
\]

What does this look like?

(iv) Given part (iii), suggest an argument that would prove

\[
\frac{1}{2\pi i} \int_{\Re(z)=c} \tilde{\varphi}(z)t^{-z}dz = \varphi(t)
\]

where the integral is over the vertical line from $c - i\infty$ to $c + i\infty$ with $c > 0$.

Solution: (i). Since $\varphi$ is smooth on $\mathbb{R}$ it is bounded in a neighbourhood of $t = 0$: $|\varphi(t)| \leq C$ for $t \in [0, t_0]$, say. Then

\[
\int_0^{t_0} |\varphi(t)t^{z-1}|dt \leq C \int_0^{t_0} t^{\Re(z)-1}dt = C t_0^{\Re(z)}/\Re(z) < \infty.
\]
Also, since \( \varphi \) is of rapid decay, for any \( z \) there exists a \( t_1 = t_1(\Re(z)) \) such that 
\[
|\varphi(t)t^{z-1}| \leq Dt^{-2} \quad \text{for all } t \geq t_1.
\]
Then
\[
\int_{t_1}^{\infty} |\varphi(t)t^{z-1}| dt \leq D/t_1 < \infty.
\]
Since the remaining integral \( \int_{t_1}^{t_0} |\varphi(t)t^{z-1}| dt \) is finite the integral is absolutely convergent in the region \( \Re(z) > 0 \).

Let \( \gamma \) be a closed curve in the region \( \Re(z) > 0 \). By absolute convergence and Fubini’s Theorem
\[
\int_{\gamma} \varphi(z) dz = \int_{\gamma} \int_0^{\infty} \varphi(t)t^{z-1} dtdz = \int_0^{\infty} \varphi(t)t^{-1} \int_{\gamma} t^z dz dt.
\]
But since \( t^z \) is analytic in the region \( \Re(z) > 0 \) for all \( t \in [0, \infty) \) this last inner integral is zero by Cauchy’s Theorem. Hence,
\[
\int_{\gamma} \varphi(z) dz = 0.
\]
The continuity of \( \varphi(z) \) follows exactly as it does for the Gamma function, hence it is analytic by Morera’s Theorem.

(ii). Integrating by parts once gives
\[
\varphi(z) = \varphi(t)t^z \bigg|_{t=0}^{\infty} - \frac{1}{z} \int_0^{\infty} \varphi'(t)t^z dt = -\frac{1}{z} \int_0^{\infty} \varphi'(t)t^z dt.
\]
By the same reasoning used in part (i), the integral \( \int_0^{\infty} \varphi'(t)t^z dt \) is analytic in the region \( \Re(z) > -1 \). Hence, the above expression provides a meromorphic continuation of \( \varphi(z) \) to said region. At \( z = 0 \) we have a residue of
\[
z \cdot -\frac{1}{z} \int_0^{\infty} \varphi'(t)t^z dt \bigg|_{z=0} = -\int_0^{\infty} \varphi'(t) dt = \varphi(0).
\]
In particular, if \( \varphi(0) = 0 \) then there is no pole at \( z = 0 \).

More generally, integrating by parts \( n \) times gives
\[
(1) \quad \varphi(z) = \frac{(-1)^n}{z(z+1) \cdots (z+n-1)} \int_0^{\infty} \varphi^{(n)}(t)t^{z+n-1} dt.
\]
Again, this expression is seen to be analytic in the region \( \Re(z) > -n \), with the exception of possible simple poles at \( z = 0, -1, \ldots, -n + 1 \). Since \( n \) is arbitrary, we’re done.
(iii). From equation (1) we see that

\[
\text{res}(\tilde{\varphi}(z), z = -n) = (z + n) \frac{(-1)^{n+1}}{z(z + 1) \cdots (z + n)} \int_0^\infty \varphi^{(n+1)}(t) t^{z+n} dt \bigg|_{z=-n}
\]

\[
= \frac{(-1)^{n+1}}{(-n)(-n+1) \cdots (-1)} \int_0^\infty \varphi^{(n+1)}(t) dt
\]

\[
= -\frac{1}{n!} \int_0^\infty \varphi^{(n+1)}(t) dt
\]

\[
= \frac{1}{n!} \varphi^{(n)}(0).
\]

In particular, if \( \varphi(t) \) is of rapid decay at zero as well as at infinity, then \( \tilde{\varphi}(z) \) is entire.

Since all the poles are simple, an extra factor of \( t^{-z} \) in the above residue calculations gives a factor of \( t^{-z}|_{z=-n} = t^n \) and hence

\[
\sum_{n=0}^{\infty} \text{res}(\tilde{\varphi}(z)t^{-z}, -n) = \sum_{n=0}^{\infty} \frac{1}{n!} \varphi^{(n)}(0) t^n.
\]

This, of course, looks like \( \varphi(t) \) in the form of a Taylor expansion about \( t = 0 \).

(iv). Assuming that the above series converges and equals \( \varphi(t) \) (so \( \varphi \) is analytic in the real analysis sense) then we would like a contour integral which captures all these residues. As suggested, we should look at

\[
\frac{1}{2\pi i} \int_{\Re(z) = c} \tilde{\varphi}(z) t^{-z} dz.
\]

We would like to truncate this integral at heights \( z = c \pm iT \) and then consider the truncated integral as part of a rectangular contour which encloses some of the residues. To estimate these integrals we need bounds on \( \tilde{\varphi}(z) \).

From equation (1) we see that for fixed \( \Re(z) \), \( \tilde{\varphi}(z) \to 0 \) as \( |\Im(z)| \to \infty \). More precisely, as \( \Im(z) \to \infty \) equation (1) gives

\[
|\tilde{\varphi}(z)| \leq \frac{1}{|z(z + 1) \cdots (z + n - 1)|} \int_0^\infty |\varphi^{(n)}(t)| t^{\Re(z)+n-1} dt \leq \frac{C_{\Re(z),n}}{|\Im(z)|^n},
\]

for some constant \( C_{\Re(z),n} \). Consequently,

\[
\frac{1}{2\pi i} \int_{\Re(z) = c} \tilde{\varphi}(z) t^{-z} dz = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \tilde{\varphi}(z) t^{-z} dz + \frac{1}{2\pi i} \left[ \int_{T}^{\infty} + \int_{-\infty}^{-T} \right] \tilde{\varphi}(c+iy) t^{-c-iy} dy
\]

\[
= \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \tilde{\varphi}(z) t^{-z} dz + O(T^{-n+1})
\]
We now consider this last integral as part of a rectangular contour $\Gamma$ whose left edge crosses the real axis at $-N - 1/2$. Then,

$$
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \hat{\varphi}(z)t^{-z}dz = \frac{1}{2\pi i} \int_{\Gamma} \hat{\varphi}(z)t^{-z}dz - \frac{1}{2\pi i} \int_{\text{other edges of } \Gamma} \hat{\varphi}(z)t^{-z}dz
$$

$$
= \sum_{n=0}^{N} \frac{1}{n!} \varphi^{(n)}(0)t^n - \frac{1}{2\pi i} \int_{\text{other edges of } \Gamma} \hat{\varphi}(z)t^{-z}dz
$$

by the residue theorem. Using our bound on $|\hat{\varphi}(z)|$ we can estimate the remaining integrals. The largest contribution comes from the integral over the left edge of $\Gamma$, and this is seen to be $O(t^{M+1/2})$. The other integrals are small in size. On letting $T \to \infty$ we get

$$
\frac{1}{2\pi i} \int_{\Re(z)=c} \hat{\varphi}(z)t^{-z}dz = \sum_{n=0}^{N} \frac{1}{n!} \varphi^{(n)}(0)t^n + O(t^{N+1/2}) = \varphi(t).
$$