COMPLEX ANALYSIS: SOLUTIONS 4

1. (i) Use Cauchy's integral formula for derivatives to compute

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{e^z}{z^{n+1}}dz, \qquad r>0.$$

(ii) Use part (i) along with Cauchy's estimate to prove that $n! \geqslant n^n e^{-n}$.

Solution: (i) From Cauchy's integral formula we have

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{e^z}{z^{n+1}}dz = \frac{1}{n!}\frac{d^n}{dz^n}(e^z)\bigg|_{z=0} = \frac{1}{n!}.$$

(ii) By Cauchy's estimate

$$\frac{1}{n!} \leqslant \frac{1}{r^n}\max_{|z|=r}|e^z| = \frac{e^r}{r^n}$$

for $r>0$. Now, to make this inequality as sharp as possible we seek to minimise the right hand side. By calculus, the function e^r/r^n has a minimum at $r=n$ where it attains the value $e^n n^{-n}$. Thus,

$$\frac{1}{n!} \leqslant e^n n^{-n}.$$

Rearranging things gives the desired inequality.

2. (i) Let $m \geqslant n$. Use Cauchy's integral formula for derivatives to compute

$$\frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{(z-1)^{n+1}}dz, \qquad r>0.$$

(ii) Prove that $\binom{m}{n} \leqslant m^m n^{-n}/(m-n)^{m-n}$.
(iii) Give a complex anlytic proof of the identity $\sum_{n=0}^m \binom{m}{n}^2 = \binom{2m}{m}$.

Solution: (i) We have

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{(z-1)^{n+1}}dz =&\frac{1}{n!}\frac{d^n}{dz^n}(z^m)\bigg|_{z=1}\\ =&\frac{1}{n!}m(m-1)\cdots(m-n+1)\\ =&\frac{1}{n!}\frac{m!}{(m-n)!}\\ =&\binom{m}{n}.\end{aligned}$$

(ii) By Cauchy's estimate we have

$$\binom{m}{n} \leqslant \frac{1}{r^n}\max_{|z-1|=r}|z^m| = \frac{(r+1)^m}{r^n}.$$

Now,

$$\frac{d}{dr}\frac{(r+1)^m}{r^n} = m(r+1)^{m-1}r^{-n} - nr^{-n-1}(r+1)^m = 0$$

iff

$$mr = n(r+1)$$

iff

$$r = \frac{n}{m-n}.$$

This choice of r minimises the right hand side of the inequality and we get

$$\binom{m}{n} \leqslant \frac{(m/(m-n))^m}{(n/(m-n))^n} = \frac{m^m n^{-n}}{(m-n)^{m-n}}$$

as desired.

(iii) Using the integral from part (i) we have

$$\begin{aligned}
\sum_{n=0}^{m}\binom{m}{n}^2 &= \sum_{n=0}^{m}\binom{m}{n}\frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{(z-1)^{n+1}}dz\\
&= \frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{z-1}\left[\sum_{n=0}^{m}\binom{m}{n}\frac{1}{(z-1)^n}\right]dz\\
&= \frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{z-1}\left(\frac{1}{z-1}+1\right)^m dz\\
&= \frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{z-1}\left(\frac{z}{z-1}\right)^m dz\\
&= \frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^{2m}}{(z-1)^{m+1}}dz\\
&= \binom{2m}{m}.
\end{aligned}$$

3. If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leqslant 1/(1 - |z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's estimate will yield.

Solution: For $r < 1$ Cauchy's estimate gives

$$|f^{(n)}(0)| \leqslant \frac{n!}{r^n}\max_{|z|=r}|f(z)| \leqslant \frac{n!}{r^n(1-r)}$$

where in the second inequality we have applied $|f(z)| \leqslant 1/(1 - |z|)$. How do we know we could not do better in this second inequality? Well, for example function $f(z) = 1/(1-z)$ satisfies the hypotheses and gives equality: $|f(z)| = 1/(1-|z|)$ when $z \in [0, 1)$. So for a general function satisfying the hypotheses, the second inequality is sharp.

As usual, we now minimise the right hand side.

$$\frac{d}{dr}\frac{1}{r^n(1-r)} = \frac{1}{r^n(1-r)^2} - \frac{n}{r^{n+1}(1-r)} = 0$$

iff

$$r = n(1-r)$$

iff

$$r = n/(n+1).$$

This choice of r gives

$$|f^{(n)}(0)| \leqslant \frac{n!}{(n/(n+1))^n(1/(n+1))} = (n+1)!\Big(1+\frac{1}{n}\Big)^n.$$

With the choice of $f(z) = 1/(1-z)$ we have $f^{(n)}(0) = n!$ so Cauchy's estimate is not very sharp here. Where do we lose out then? (...recall Cauchy's estimate is essentially a consequence of the estimation lemma).

4. Find the maxima of $f(z) = z^2 - 1$ on the closed disk $|z| \leqslant 1$.

Solution: By the maximum modulus principle we know that the maximum of $|f(z)|$ must occur on the boundary $|z| = 1$. So, we can set $z = e^{i\theta}$, $\theta \in [0, 2\pi]$, and consider

$$|f(z)|^2 = |e^{2i\theta} - 1|^2 = (e^{2i\theta} - 1)(e^{-2i\theta} - 1) = 2 - e^{2i\theta} - e^{-2i\theta} = 2 - 2\cos 2\theta.$$

The maxima of this function occur at $\theta = \pi/2, 3\pi/2$, i.e. at $z = i, -i$ and at these points we have $|f(z)| = 2$.

5. Determine the analytic continuations from $|z| < 1$ to as large a region as possible of the following power series

$$\sum_{n=1}^{\infty} (-1)^n n z^{n-1}, \qquad \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Solution: For $|z| < 1$ We have

$$\sum_{n=1}^{\infty} (-1)^n n z^{n-1} = -\frac{1}{(1+z)^2}.$$

Therefore, the function on the right provides the analytic continuation of the series from $|z| < 1$ to $\mathbb{C}\backslash -1$. Similarly, for $|z| < 1$

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$$

and so this provides the analytic continuation to $\mathbb{C}\backslash\mathbb{R}_{\geqslant 1}$.

6. Find and classify all singularties of the function $f(z) = e^{1/(z-i)} \tan z$. Determine the order of any poles.

Solution: At $z = i$ we have an essential singularity. The function $\tan z$ is meromorphic with poles at the zeros of $\cos z$ i.e. at $z_n = \pi/2 + n\pi$ for $n \in \mathbb{Z}$. About these points $\cos z$ has Taylor series

$$\begin{aligned}\cos z &= \cos z_n - (z - z_n)\sin z_n - \frac{(z-z_n)^2}{2!}\cos z_n + \frac{(z-z_n)^3}{3!}\sin z_n + \cdots \\ &= -(z - z_n)(\sin z_n)\Big(1 - \frac{(z-z_n)^2}{3!} + \cdots\Big).\end{aligned}$$

So the zeros at z_n are of order 1 and so the corresponding poles are all simple. Indeed,

$$\lim_{z\to z_n}(z-z_n)\tan z = \sin z_n \lim_{z\to z_n}\frac{z-z_n}{\cos z} = -\sin z_n\cdot\frac{1}{\sin z_n} = -1$$

which exists. Incidentally, this shows that the residue at all poles is -1.

7. Let $f : G\backslash\{z_0\} \to \mathbb{C}$ be analytic with a removable singularity at z_0. Show that

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

for all $z \in G\backslash\{z_0\}$ and all positively oriented contours $\gamma \subset G\backslash\{z_0\}$ enclosing z i.e. Cauchy's formula still holds. What can be said if $z = z_0$?

Solution: If γ does not enclose z_0 then f is analytic on the interior of the contour and so Cauchy's formula clearly still holds. If γ encloses z_0 then since we're assuming $z \neq z_0$ we may surround both z_0 and z by non-intersecting open disks, $D_{r_0}(z_0)$ and $D_r(z)$ say, both of which are contained in the interior of γ.

Then, by the usual deformation trick with Cauchy's theorem we have

$$\int_\gamma \frac{f(w)}{w-z}dw = \int_{\partial D_{r_0}(z_0)}\frac{f(w)}{w-z}dw + \int_{\partial D_r(z)}\frac{f(w)}{w-z}dw$$

since the integrand is analytic on the area between the contours. By Cauchy's integral formula this second integral is given by $f(w)$, so we only need show that the first integral vanishes.

Since the disks are non-intersecting the term $w-z$ is bounded away from zero; $|w-z| > c > 0$. Also, since f has a removable singularity at z_0 there exists an analytic continuation g such that $f(z) = g(z)$ for all $0 < |z-z_0| < r_0$ and consequently $|f(z)|$ is bounded in this region: $|f(z)| < M$. Then, by the estimation lemma

$$\left|\int_{\partial D_{r_0}(z_0)}\frac{f(w)}{w-z}dw\right| \leqslant 2\pi r_0 M/c.$$

Since r_0 was arbitrary we can let it tend to zero to give the desired result. Alternatively, one could use the condition $\lim_{z\to z_0}(z-z_0)f(z) = 0$ and proceed as in the proof of Cauchy's integral formula.

If $z = z_0$ then the left hand side is undefined. However, the integral on the right

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0}dw$$

does make sense: f is analytic and hence continuous on the contour γ so the integral exists. In fact, this integral is an analytic function for all $z \in G$ (why?) and hence represents the analytic continuation of f from $G\backslash\{z_0\}$ to G.