COMPLEX ANALYSIS: PROBLEMS 5

DUE FRIDAY 8TH APRIL

1. Find the poles and residues of the following functions

$$\frac{1}{z^4+5z^2+6}, \qquad \frac{1}{(z^2-1)^2}, \qquad \frac{\pi\cot(\pi z)}{z^2}, \qquad \frac{1}{z^m(1-z)^n} \quad (m,n\in\mathbb{Z}_{>0})$$

2. Use the substitution $e^{i\theta}=z$ along with the residue theorem to show that

$$\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=\frac{2\pi}{\sqrt{3}}.$$

3. Evaluate the following integrals via residues. Show all estimates.

(i)

$$\int_0^\infty \frac{x^2}{x^4+5x^2+6}dx$$

(ii)

$$\int_0^\infty \frac{x\sin x}{x^2+a^2}dx, \quad a \text{ real}$$

(iii)

$$\int_0^\infty \frac{\log x}{1+x^2}dx$$

4. Let Γ_N be the square which crosses the real axis at $\pm(N+1/2)$ with $N\in\mathbb{N}$.

(i) Show that $\cot(\pi z)$ is bounded on Γ_N and hence show that

$$\lim_{N\to\infty}\int_{\Gamma_N}\frac{\pi\cot(\pi z)}{z^2}dz=0.$$

(ii) For a given N compute the above integral via residues. Conclude something interesting.

5. (i) Let $f(z)=z^6+\cos z$. Find the change in argument of $f(z)$ as z travels once around the circle of radius 2, center zero, in the positive direction.

(ii) How many solutions does $3e^z-z=0$ have in the disk $|z|\leqslant 1$?

(iii) Use Rouche's Theorem to prove that a polynomial of degree n has n roots in $\mathbb{C}$.

6. Prove that the function

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(n+z)^2}$$

is meromorphic on $\mathbb{C}$.

7. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth function whose derivatives $\varphi^{(k)}(t)$, $k \geqslant 0$, are of rapid decay at ∞ i.e.

$$\lim_{t \to \infty} t^A \varphi^{(k)}(t) = 0$$

for all $A \in \mathbb{R}$ and all $k \geqslant 0$. The *Mellin transform* of φ is defined as

$$\tilde{\varphi}(z) = \int_0^\infty \varphi(t) t^{z-1} dt.$$

(i) Prove that $\tilde{\varphi}(z)$ is analytic in the region $\Re(z) > 0$.

(ii) Use integration by parts to show that $\tilde{\varphi}(z)$ can be analytically continued to $\mathbb{C} \backslash \mathbb{Z}_{\leqslant 0}$, with possible simple poles at $z = -n$.

(iii) Find the residues at these poles and compute the formal sum

$$\sum_{n=0}^{\infty} \operatorname{res}(\tilde{\varphi}(z) t^{-z}, -n).$$

What does this look like?

(iv) Given part (iii), suggest an argument that would prove

$$\frac{1}{2\pi i} \int_{\Re(z)=c} \tilde{\varphi}(z) t^{-z} dz = \varphi(t)$$

where the integral is over the vertical line from $c - i\infty$ to $c + i\infty$ with $c > 0$.