COMPLEX ANALYSIS: PROBLEMS 4

DUE FRIDAY 11TH FEBRUARY

1. (i) Use Cauchy's integral formula for derivatives to compute

$$\frac{1}{2\pi i}\int_{|z|=r}\frac{e^z}{z^{n+1}}dz, \qquad r>0.$$

(ii) Use part (i) along with Cauchy's estimate to prove that $n! \geqslant n^n e^{-n}$.

2. (i) Let $m \geqslant n$. Use Cauchy's integral formula for derivatives to compute

$$\frac{1}{2\pi i}\int_{|z-1|=r}\frac{z^m}{(z-1)^{n+1}}dz, \qquad r>0.$$

(ii) Prove that $\binom{m}{n} \leqslant m^m n^{-n}/(m-n)^{m-n}$.
(iii) Give a complex anlytic proof of the identity $\sum_{n=0}^{m}\binom{m}{n}^2 = \binom{2m}{m}$.

3. If $f(z)$ is analytic for $|z|<1$ and $|f(z)| \leqslant 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's estimate will yield.

4. Find the maxima of $f(z) = z^2 - 1$ on the closed disk $|z| \leqslant 1$.

5. Determine the analytic continuations from $|z|<1$ to as large a region as possible of the following power series

$$\sum_{n=1}^{\infty}(-1)^n n z^{n-1}, \qquad \sum_{n=1}^{\infty}\frac{z^n}{n}.$$

6. Find and classify all singularties of the function $f(z) = e^{1/(z-i)}\tan z$. Determine the order of any poles.

7. Let $f : G\backslash\{z_0\} \to \mathbb{C}$ be analytic with a removable singularity at z_0. Show that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(w)}{w-z}dw$$

for all $z \in G\backslash\{z_0\}$ and all positively oriented contours $\gamma \subset G\backslash\{z_0\}$ enclosing z i.e. Cauchy's formula still holds. What can be said if $z = z_0$?