Project 2, Complex Analysis
due Monday March 31, 2014

Problem 1:
Let $f(z) = \frac{(\cos z - 1) e^{\frac{i}{z-1}}}{z^3(3z^2-27)}$.
a) Find the singularities of f and characterize them.
b) Find the integral $\oint_{|z|=5} f(z)dz$

Solution, Problem 1:

$$f(z) = \frac{\cos z - 1}{z^3} \frac{1}{3(z-3)(z+3)} e^{\frac{i}{z-1}}.$$

a) We can see that $z_1 = 1$ is an essential singularity since

$$e^{\frac{i}{z-1}} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n \left(\frac{1}{z-1} \right)^n.$$

At $z_2 = 3$ or $z_3 = -3$ we have simple poles. Near $z_4 = 0$ we see that

$$\begin{aligned} \frac{\cos z - 1}{z^3} &= \frac{-\frac{1}{2}z^2 + \frac{1}{4!}z^4 - \cdots}{z^3} \\ &= -\frac{1}{2}\frac{1}{z} + \frac{1}{4!}z - \cdots \end{aligned}$$

so $z = 0$ is a simple pole.

b)

$$\oint_{|z|=5} f(z)dz = 2\pi i \sum_{j=1}^{4} \mathrm{Res}_{z=z_j} f(z)$$

since z_1, z_2, z_3, z_4 are all inside the disc of radius 5.

Now

$$\begin{aligned}
\mathrm{Res}_{z=z_4} f(z) &= \lim_{z\to 0} z f(z) \\
&= (-1/2)\frac{1}{3(-3)(3)} e^{i} \\
&= e^{-i}/54. \\
\mathrm{Res}_{z=z_3} f(z) &= \lim_{z\to -3} (z+3) f(z) \\
&= \frac{\cos(-3)-1}{(-3)^3}\frac{1}{3(-3-3)} e^{\frac{i}{-3-1}} \\
&= \frac{1}{3^5}\frac{\cos 3 - 1}{2} e^{-i/4} \\
\mathrm{Res}_{z=z_2} f(z) &= \lim_{z\to 3} (z-3) f(z) \\
&= \frac{\cos 3 - 1}{3^3}\frac{1}{3+3} e^{\frac{1}{3-1}} \\
&= \frac{1}{3^5}\frac{1}{2}(\cos 3 - 1) e^{i/2}
\end{aligned}$$

Finally at $z = 1$ things are more complicated.

$$\begin{aligned}
f(z) &= g(z) e^{\frac{i}{z-1}} \\
&= g(z)\sum_{n=0}^{\infty}\frac{1}{n!} i^n \frac{1}{(z-1)^n} \\
&= \left(\sum_{k=0}^{\infty}\frac{g^{(k)}(1)}{k!}(z-1)^k\right)\left(\sum_{n=0}^{\infty}\frac{1}{n!} i^n \frac{1}{(z-1)^n}\right)
\end{aligned}$$

Notice that when $n \geq 1$ and $k = n-1$, then

$$\begin{aligned}
\frac{g^{(k)}(1)}{k!}(z-1)^k \frac{1}{n!} i^n \frac{1}{(z-1)^n} &= \frac{g^{(n-1)}(1)}{(n-1)!}(z-1)^{n-1}\frac{1}{n!} i^n \frac{1}{(z-1)^n} \\
&= \frac{g^{(n-1)}(1)}{n!(n-1)!} i^n \frac{1}{(z-1)}
\end{aligned}$$

so $\mathrm{Res}_{z=z_1} f(z) = \sum_{n=1}^{\infty} \frac{g^{(n-1)}(1)}{n!(n-1)!} i^n$

Problem 2:
Let $h(z)$ be analytic in $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. Suppose that $h(0) = h'(0) = h''(0) = 0$. Assume that $|h| < 1$.
a) Show that $|h(z)| \leq |z|^3$.
b) Show that if $h(1/2) = 1/8$, then $h((\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2})\frac{1}{8}) = e^{3i\pi/4}/2^9$.

Solution Problem 2:

a) Notice that since $h(0) = h'(0) = h''(0) = 0$, then $g(z) = \frac{h(z)}{z^3}$ has a removable singularity. Also

$$|g(z)| = \frac{|h(z)|}{|z^3|} \leq \frac{1}{|z^3|}.$$

Let $|z| = 1 - \epsilon$, then $|g(z)| \leq \frac{1}{(1-\epsilon)^3}$ so by the maximum principle $|g(z)| \leq \frac{1}{(1-\epsilon)^3}$ for all z such that $|z| \leq 1 - \epsilon$. Let $\epsilon \to 0$. Then we get $|g(z)| \leq 1$ so $|h(z)| \leq |z|^3$.

b) If $h(1/2) = 1/8$ it follows that $g(1/2) = 1$, hence g takes a maximum at an interior point, so it must be constant. This implies that $h(z) = z^3$ and therefore

$$\begin{aligned} h((\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\frac{1}{8}) &= h(e^{i\pi/4}\frac{1}{8}) \\ &= (e^{i\pi/4}\frac{1}{8})^3 \\ &= e^{3i\pi/4}(1/2^3)^3 \\ &= e^{3i\pi/4}\frac{1}{2^9} \end{aligned}$$

Problem 3:
a) Show that $10z^5 + e^z + \cos z$ has exactly 5 zeroes in the unit disc
$\Delta = \{|z| < 1\}$.
b) Show that

$$\frac{1}{2\pi i}\oint_{|z|=1} \frac{50z^4 + e^z - \sin z}{10z^5 + e^z + \cos z} dz = 5.$$

Solution Problem 3:
a) Let $f(z) = 10z^5$ and $g(z) = e^z + \cos z$. Then $|f(z)| = 10$ when $|z| = 1$ while

$$\begin{aligned} |g(z)| &= |e^z + \frac{1}{2}(e^{iz} + e^{-iz})| \\ &\leq |e^z| + \frac{1}{2}|e^{iz}| + \frac{1}{2}|e^{-iz}|. \end{aligned}$$

When $|z| = 1$, it follows that $-1 \leq \text{Re}(z) \leq 1$ and $-1 \leq \text{Im}(z) \leq 1$ so

$$\begin{aligned} |e^z| + \frac{1}{2}|e^{iz}| + \frac{1}{2}|e^{-iz}| &= e^{\text{Re}(z)} + \frac{1}{2}e^{-\text{Im}(z)} + \frac{1}{2}e^{\text{Im}(z)} \\ &\leq e + \frac{1}{2}e + \frac{1}{2}e \\ &= 2e \\ &< 10. \end{aligned}$$

Hence by Rouche's theorem it follows that $f(z) + g(z)$ has as many 0's in the unit disc as f does, and f has 5 zeroes at 0.

b) Let $h(z) = 10z^5 + e^z + \cos z$. Then $h'(z) = 50z^4 + e^z - \sin z$. So

$$\begin{aligned}\frac{1}{2\pi i}\oint \frac{50z^4 + e^z - \sin z}{10z^5 + e^z + \cos z}dz &= \frac{1}{2\pi i}\oint_{|z|=1} \frac{h'(z)}{h(z)}dz \\ &= N_0 - N_\infty\end{aligned}$$

where N_0 is the number of 0's for h in the unit disc and N_∞ is the number of poles. So $N_0 - N_\infty = 5 - 0 = 5$.

Problem 4:
Find

$$\int_0^\infty \frac{x^{3/2}}{(1+x)^4}dx.$$

Show all estimates.

Solution of Problem 4:

(1) Notice that $x + 1 \geq 1$ when $x \geq 0$ so the function is continuous in $[0, \infty)$. Also

$$\int_0^\infty \frac{x^{3/2}}{(1+x)^4}dx = \int_0^1 \frac{x^{3/2}}{(1+x)^4}dx + \int_1^\infty \frac{x^{3/2}}{(1+x)^4}dx.$$

The first integral converges since the integrand is continuous. Now

$$0 \leq \frac{x^{3/2}}{(1+x)^4} \leq \frac{x^{3/2}}{x^4} = x^{-5/2}$$

when $1 \leq x < \infty$ and

$$\int_1^\infty x^{-5/2}dx < \infty$$

so the second integral converges also.

(2) Let $T_{\epsilon,R} = C_R \cup L_1 \cup L_2 \cup \gamma_\epsilon$ where C_R is the circle around 0 of radius R traversed counter-clockwise, $L_1 = [\epsilon, R], L_2 = [R, \epsilon]$ and γ_ϵ is the circle around 0 of radius ϵ traversed clockwise. Now

$$\int_{T_{\epsilon,R}} \frac{z^{3/2}}{(1+z)^4}dz = 2\pi i \mathrm{Res}_{z=-1}\frac{z^{3/2}}{(1+z)^4}$$

when $\epsilon < 1$ and $R > 1$ $\frac{z^{3/2}}{(1+z)^4}$ has a pole of order 4 at $z = -1$ and so

$$\begin{aligned}
\mathrm{Res}_{z=-1} f(z) &= \lim_{z\to -1} \frac{1}{3!}\frac{d^3}{dz^3}(z+1)^4 f(z)) \\
&= \frac{1}{3!}\lim_{z\to -1}\frac{d^3}{dz^3}(z^{3/2}) \\
&= \frac{1}{3!}\lim_{z\to -1}\frac{3}{2}\frac{1}{2}\frac{-1}{2}z^{-3/2} \\
&= \frac{1}{3\cdot 2}\frac{3}{8}(-1)\left(e^{i\pi}\right)^{3/2} \\
&= -\frac{1}{16}e^{3i\pi/2} \\
&= -i/16
\end{aligned}$$

So $\int_{T_{\epsilon,R}} \frac{z^{3/2}}{(1+z)^4} dz = 2\pi i(i/16) = \pi/8.$

Now

$$\begin{aligned}
\left|\int_{C_R} \frac{z^{3/2}}{(1+z)^4} dz\right| &\leq \int_{C_R} \frac{|z|^{3/2}}{|1+z|^4}|dz| \\
&\leq \frac{R^{3/2}}{(R-1)^4} 2\pi R \\
&\to 0
\end{aligned}$$

when $R \to \infty$.
Also

$$\begin{aligned}
\left|\int_{\gamma_\epsilon} \frac{z^{3/2}}{(1+z)^4} dz\right| &\leq \int_{\gamma_\epsilon} \frac{|z|^{3/2}}{(1-|z|)^4}|dz| \\
&= \frac{1}{(1-\epsilon)^4}\epsilon^{5/2}\pi\cdot 2 \\
&\to 0
\end{aligned}$$

when $\epsilon \to 0$.
Morever

$$\begin{aligned}
\int_{L_2} \frac{z^{3/2}}{(1+z)^4} dz &= \int_R^\epsilon \frac{x^{3/2}e^{\frac{3}{2}2\pi i}}{(1+x)^4} dx \\
&= -\int_\epsilon^R \frac{x^{3/2}}{(1+x)^4} e^{3\pi i} dx \\
&= \int_\epsilon^R \frac{x^{3/2}}{(1+x)^4} dx
\end{aligned}$$

So

$$\begin{aligned}
\lim_{\epsilon\to 0, R\to\infty}\int_{T_{\epsilon,R}}\frac{z^{3/2}}{(1+z)^4}dz &= \lim\int_{L_1}\frac{z^{3/2}}{(1+z)^4}dz \lim\int_{L_2}\frac{z^{3/2}}{(1+z)^4}dz \\
&= \int_0^\infty \frac{x^{3/2}}{(1+x)^4}dx + \int_0^\infty \frac{x^{3/2}}{(1+x)^4}dx \\
&= 2\int_0^\infty \frac{x^{3/2}}{(1+x)^4}dx \\
&= \pi/8
\end{aligned}$$

Hence

$$\int_0^\infty \frac{x^{3/2}}{(1+x)^4}dx = \pi/16.$$

Problem 5:
a) Find the radius of convergence of $\sum_{n=1}^{\infty}\left(\frac{i(n^2+3)}{3(2n^2-1)}\right)^n z^n$.
b) Find out where the function

$$g(z) = \sum_{n=1}^{\infty}\left(\frac{i(n^2+3)}{3(2n^2-1)}\right)^n (z^n + \frac{1}{z^n})$$

is analytic.

Solution Problem 5:
a) $\sum_{n=1}^{\infty}\left(\frac{i(n^2+3)}{3(2n^2-1)}\right)^n z^n = \sum_{n=1}^{\infty} a_n z^n$. The radius of convergence R for this series, is found as

$$\begin{aligned}
\frac{1}{R} &= \lim_{n\to\infty} |a_n|^{1/n} \\
&= \lim_{n\to\infty} \frac{i(n^2+3)}{3(2n^2-1)} \\
&= 1/6
\end{aligned}$$

Hence $R = 6$.

b)

$$g(z) = \sum_{n=1}^{\infty} a_n(z^n + \frac{1}{z^n}) = \sum_{n=1}^{\infty} a_n(z^n) + \sum_{n=1}^{\infty} a_n(\frac{1}{z^n})$$

The first sum converges when $|z| < 6$, the second converges when $|1/z| < 6$ or $|z| > 1/6$ so both converge when $1/6 < |z| < 6$.