Project 1, Complex Analysis
due Monday February 10, 2014

Problem 1: Let $D = \{z \in \mathbb{C}; 0 < \mathrm{Im}(z) < \frac{\pi}{2}\}$.
a) Show that $f(z) = e^z$ maps D onto $A = \{z \in \mathbb{C}; \mathrm{Re}(z) \text{ and } \mathrm{Im}(z) > 0\}$.
b) Show that $z \to z^2$ maps A onto $H^+ = \{z; \mathrm{Im}(z) > 0\}$.
c) Use a) and b) to find a map from D onto the unit disc $\Delta = \{z \in \mathbb{C}; |z| < 1\}$.

Solution Problem 1. $D = \{z \in \mathbb{C}; 0 < \mathrm{Im}(z) < \frac{\pi}{2}\}$. Observe that D is made out of lines $L_y = \{x + iy, -\infty < x < \infty, 0 < y < \pi/2\}$.
a) The map $f(z) = e^z = e^{x+iy} = e^x e^{iy}$ sends a line L_y to a ray $R_y = \{w = e^z, 0 < |w| = e^x < \infty, \arg(w) = y\}$. As y goes from 0 to $\pi/2$, the rays R_y sweeps out the region $A = \{z \in \mathbb{C}; \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\} = \{z \in \mathbb{C}; 0 < \arg(z) < \pi/2\}$.
b) Observe that the map $z \to z^2$ sends a point $z = re^{i\theta}$ to the point $w = r^2 e^{2i\theta}$, hence the set $A = \{0 < \arg(z) < \pi/2\}$ will be sent to the region $H^+ = \{w; 0 < \mathrm{Im}(w) > 0\}$.
c) First we shall look for a map of the kind $T(z) = \frac{az+b}{cz+d}$ that will send H^+ to the unit disc $\Delta = \{z \in \mathbb{C}; [z[< 1.\}$. What we need is for T to send the boundary of $H^+, L = \{x + i0, -\infty < x < \infty\}$, to the boundary of Δ, $S = \{z; |z| = 1\}$ and the interior of H^+ to the interior of Δ.
Let us first start with the first task. The way to send L to S is to choose 3 points on L and send them to three points on S. For example,

$$\begin{aligned} -1 &\to -1 \\ 0 &\to i \\ 1 &\to 1 \end{aligned}$$

Then we get the following equations:

$$\begin{aligned} \frac{-a+b}{-c+d} &= -1 \text{ or } -a+b = c-d \\ \frac{b}{d} &= i \text{ or } b = id \\ \frac{a+b}{c+d} &= 1 \text{ or } a+b = c+d \end{aligned}$$

When we solve these equations we get $b = id, c = id$ and $a = d$ so

$$T(z) = \frac{dz + id}{idz + d} = \frac{z + i}{iz + 1} = \frac{1}{i}\frac{z + i}{z - i}.$$

Now $iT(z) = z + iz - i$ and $i = e^{i\pi/2}$ so $i(Tz)$ sends L to S. Now $iT(z)$ sends H^+ either to Δ or to $\mathbb{C} \setminus \overline{\Delta}$. We only need to test the image of one point say i. We see that $iT(i) = \infty$ so clearly H^+ is sent to $\mathbb{C} \setminus \overline{\Delta}$. To fix this we use the map $w \to 1/w$ or $z \to \frac{1}{iT(z)} = \frac{z-i}{z+i}$. Finally let us use a), b) and c) to send D to Δ. First we let $z \to e^z = w$, then D goes to A, next $w \to w^2$, $z \to e^z \to e^{2z} = \xi$ sends D to A and then to H^+. Finally $\xi \to \frac{\xi - i}{\xi + i}$ or $z \to e^z \to e^{2z} \to \frac{e^{2z}-i}{e^{2z}+i}$ which sends D to Δ.

Problem 2: Let $u(x, y) = xe^x \cos y - ye^x \sin y$
a) Show that u is harmonic in the entire plane.
b) Find a function v so that $f = u + iv$ is analytic.
c) Let $h(\theta) = \cos\theta e^{\cos\theta} \cos(\sin\theta) - \sin\theta e^{\cos\theta} \sin(\sin\theta)$ and show that $\int_0^{2\pi} h(\theta) = 0$. (Hint: Observe that $u(\cos\theta, \sin\theta) = h(\theta)$.)

Solution Problem 2:

$$u(x, y) = xe^x \cos y - ye^x \sin y.$$

a) u is harmonic if $u_{xx} + u_{yy} = 0$. Now

$$\begin{aligned}
u_x &= e^x \cos y + xe^x \cos y - ye^x \sin y \\
u_{xx} &= e^x \cos y + e^x \cos y + xe^x \cos y - ye^x \sin y \\
&= 2e^x \cos y + xe^x \cos y - ye^x \sin y \\
&\text{and} \\
u_y &= -xe^x \sin y - e^x \sin y - ye^x \cos y \\
u_{yy} &= -xe^x \cos y - e^x \cos y - e^x \cos y + ye^x \sin y \\
&= -2e^x \cos y - xe^x \cos y + ye^x \sin y
\end{aligned}$$

So, $u_{xx} + u_{yy} = 0$ in the entire plane.

b) If $u + iv = f$ is entire, we need to u to satisfy the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y \\ u_y &= -y_x \end{aligned}$$

so

$$\begin{aligned} v_y &= e^x \cos y + xe^x \cos y - ye^x \sin y \\ &\text{and} \\ v_x &= xe^x \sin y + e^x \sin y + ye^x \cos y \end{aligned}$$

From this we learn that

$$\begin{aligned} v &= e^x \sin y + xe^x \sin y + ye^x \cos y - e^x \sin y + h(x) \\ &= xe^x \sin y + ye^x \cos y + h(x) \\ &\text{also} \\ v &= xe^x \sin y + ye^x \cos y + g(y) \end{aligned}$$

so the function $v = xe^x \sin y + ye^x \cos y$ will do.

c) The function $h(\theta) = u(0 + ie^{i\theta})$. The subaveraging theorem for harmonic functions states

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(0 + e^{i\theta}) d\theta.$$

But $u(0) = 0$, so

$$\int_0^{2\pi} u(0 + e^{i\theta}) d\theta = \int_0^{2\pi} h(\theta) d\theta = 0.$$

Problem 3: Let f be analytic in $\Delta = \{z; |z| < 1\}$ and show that if $f(z) \in \mathbb{R}$ for every $z \in \Delta$, then f is constant.

Solution Problem 3
$f = u + iv$ is analytic and $f(z) \in \mathbb{R}$ for each $z \in \Delta = \{|z| < 1\}$. This means that v is constantly equal to 0 in Δ. The Cauchy-Riemann equations $u_x = v_y, u_y = -u_x$ will force $u_x = v_y = 0$ and $u_y = -v_x = 0$ so both u and v will have to be constant.

Problem 4: Find $\oint_C \frac{\sin z}{z^3(z-1)} dz$ where C is the circle with center 0 and radius 2 travelled counterclockwise.

Solution Problem 4:

We see that $\frac{sinz}{z^3(z-1)}$ is analytic in $\{z; |z| < 2, z \neq 0, 1\}$. From Cauchy's theorem it follows that

$$\int_C \frac{sinz}{z^3(z-1)} dz = \int_{C_0} \frac{sinz}{z^3(z-1)} dz + \int_{C_1} \frac{sinz}{z^3(z-1)} dz$$

where C_0 is the circle $\{|z| = 1/3\}$ and C_1 is the circle $\{z; |z-1| = 1/3\}$, both traveled counterclockwise. The function $h(z) = \frac{\sin z}{z-1}$ is analytic in $\{z; |z| < 1/3\}$ and

$$\begin{aligned} I_0 &= \int_{C_0} \frac{\sin z}{z^3(z-1)} dz \\ &= \int_{C_0} \frac{h(z)}{z^3} dz \\ &= \int \frac{h(z)}{z^{2+1}} dz \\ &= 2\pi i \frac{1}{2} \left(\frac{2}{2\pi i} \int_{C_0} \frac{h(z)}{z^{2+1}} \right) \end{aligned}$$

Cauchy's formula for derivatives gives

$$\begin{aligned} I_1 &= 2\pi i \frac{1}{2} h''(0) \\ &= \pi i h''(0) \end{aligned}$$

Now $h'(z) = \frac{\cos z}{z-1} - \frac{\sin z}{(z-1)^2}$,

$$h''(z) = \frac{-\sin z}{(z-1)} - \frac{\cos z}{(z-1)^2} - \frac{\cos z}{(z-1)^2} + 2\frac{\sin z}{(z-1)^3},$$

so $h''(0) = -2$.
Hence $I_0 = -2\pi i$. While $g(z) = \frac{\sin z}{z^3}$ is analytic in $\{z; |z-1| \le 1/3\}$ so

$$\begin{aligned} I_1 &= \int_{C_1} \frac{\sin z}{z^3(z-1)} \\ &= \int_{C_1} \frac{g(z)}{z-1} dz \\ &= 2\pi i \left(\frac{1}{2\pi i} \int_{C_1} \frac{g(z)}{z-1} dz \right) \\ &= 2\pi i g(1) \\ &= 2\pi i \sin 1 \end{aligned}$$

from Cauchy's integral formula.
Hence

$$\begin{aligned} \oint_C \frac{\sin z}{z^3(z-1)} dz &= I_0 + I_1 \\ &= -2\pi i + 2\pi i \sin 1 \\ &= 2\pi i(-1 + \sin 1) \end{aligned}$$

Problem 5:
a) Prove that if $z \neq 1$, then $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$.
b) Use this to show that

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}$$

Solution Problem 5:

Let $s_n = 1 + z + z^2 + \cdots + z^n$.
a) We want to prove that $s_n = \frac{1-z^{n+1}}{1-z}$. Now $zs_n = z + z^2 + \cdots + z^n + z^{n+1}$ so $s_n - zs_n = 1 - z^{n+1}$ so $(1-z)s_n = 1 - z^{n+1}$. If $z = 1$, then all this is telling us is that $0 = 0$ but if $z \neq 1$, then it follows that $s_n = \frac{1-z^{n+1}}{1-z}$.
b) $s_n = 1+z+z^2+\cdots+z^n$ so $\frac{ds_n}{dz} = 1+2z+3z^2+\cdots+nz^{n-1}$. But $s_n = \frac{1-z^{n+1}}{1-z}$ so

$$
\begin{aligned}
\frac{ds_n}{dz} &= -\frac{(n+1)z^n}{1-z} + \frac{1-z^{n+1}}{(1-z)^2} \\
&= \frac{-(n+1)z^n(1-z) + 1 - z^{n+1}}{(1-z)^2} \\
&= \frac{-(n+1)z^n + (n+1)z^{n+1} + 1 - z^{n+1}}{(1-z)^2} \\
&= \frac{1 - z^n - nz^n(1-z)}{(1-z)^2} \\
&= \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}
\end{aligned}
$$