Project 1, Complex Analysis
due Monday February 10, 2014

Problem 1: Let $D = \{z \in \mathbb{C}; 0 < \mathrm{Im}(z) < \frac{\pi}{2}\}$.
a) Show that $f(z) = e^z$ maps D onto $A = \{z \in \mathbb{C}; \mathrm{Re}(z) \text{ and } \mathrm{Im}(z) > 0\}$.
b) Show that $z \to z^2$ maps A onto $H^+ = \{z; \mathrm{Im}(z) > 0\}$.
c) Use a) and b) to find a map from D onto the unit disc $\Delta = \{z \in \mathbb{C}; |z| < 1\}$.

Problem 2: Let $u(x, y) = xe^x \cos y - ye^x \sin y$
a) Show that u is harmonic in the entire plane.
b) Find a function v so that $f = u + iv$ is analytic.
c) Let $h(\theta) = \cos\theta e^{\cos\theta} \cos(\sin\theta) - \sin\theta e^{\cos\theta} \sin(\sin\theta)$ and show that $\int_0^{2\pi} h(\theta) = 0$. (Hint: Observe that $u(\cos\theta, \sin\theta) = h(\theta)$.)

Problem 3: Let f be analytic in $\Delta = \{z; |z| < 1\}$ and show that if $f(z) \in \mathbb{R}$ for every $z \in \Delta$, then f is constant.

Problem 4: Find $\oint_C \frac{\sin z}{z^3(z-1)} dz$ where C is the circle with center 0 and radius 2 travelled counterclockwise.

Problem 5:
a) Prove that if $z \neq 1$, then $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$.
b) Use this to show that

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}$$