ANSWERS TO THE FINAL EXAM
TMA 4175
COMPLEX ANALYSIS
16. MAY 2014

PROBLEM 1:

Let $D = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < \pi\}$

The map $z \to e^{i\frac{\pi}{2}} z = iz$ sends

D to $\Omega = \{w \in \mathbb{C} : 0 < \operatorname{Im} w < \pi\}$,

$w \to e^{w}$ sends Ω to $H^+ = \{\xi \in \mathbb{C} : \operatorname{Im} \xi > 0\}$

Finally $\xi \to \dfrac{\xi - i}{\xi + i}$ sends H^+ to

the unit disc $\Delta = \{\eta : |\eta| < 1\}$

All together.

$$f(z) = \frac{e^{iz} - i}{e^{iz} + i}$$

sends D to Δ.

PROBLEM 2:

$$f(z) = z^{10} + 2z^2 + \tfrac{1}{2} z$$

a) Let $u(z) = z^{10}$ and $v(z) = 2z^2 + \frac{1}{2} z$

When $|z| = 2$, $|u(z)| = 2^{10}$ while

$|v(z)| \le 2|z|^2 + \frac{1}{2}|z| = 8 + 1 = 9$ so

$|u(z)| > |v(z)|$ when $|z| = 2$

Rouches theorem implies that

$u(z)$ and $f(z) = u(z) + v(z)$ have the

same number of zero's in $\{z : |z| < 2\}$

b) When $|z|=1$ we see that
$|2z^2|=2$ while $|z^{10}+\frac{1}{2}z| \leq |z|^{10}+\frac{1}{2}|z| = \frac{3}{2}$
so $|2z^2| > |z^{10}+\frac{1}{2}z|$ when $|z|=1$
Now $2z^2$ have two zeros in $\{z : |z|<1\}$
so $2z^2+z^{10}+\frac{1}{2}z$ have two zeros
in $\{z: |z|<1\}$ also $2z^2+z^{10}+\frac{1}{2}z \neq 0$ when
$|z|=1$ hence

$f(z)$ have $10-2$ zeros in

$\{z : 1<|z|<2\}$.

PROBLEM 3:

$$f(z) = \frac{\sin z}{z^2} \frac{1}{z^2+3}$$

$$= \frac{1}{z} \frac{\sin z}{z} \frac{1}{(z-i\sqrt{3})} \frac{1}{(z+i\sqrt{3})}$$

$\lim_{z\to 0} \frac{\sin z}{z} = 1$ so f have simple
poles at $z_0=0$, $z_1=i\sqrt{3}$ and $z_3=-i\sqrt{3}$

If $\gamma(t)=2e^{it}$, then

$$\int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=i\sqrt{3}} f(z) + \operatorname{Res}_{z=-i\sqrt{3}} f(z)\right)$$

$$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{\sin z}{z} \frac{1}{z^3+3} = \frac{1}{3}$$

$$\operatorname{Res}_{z=i\sqrt{3}} f(z) = \lim_{z\to i\sqrt{3}} (z-i\sqrt{3}) f(z) = \lim_{z\to i\sqrt{3}} \frac{\sin z}{z^2} \frac{1}{z+i\sqrt{3}} = \frac{\sin(i\sqrt{3})}{-i6\sqrt{3}}$$

$$\operatorname{Res}_{z=-i\sqrt{3}} f(z) = \lim_{z\to -i\sqrt{3}} (z+i\sqrt{3}) f(z) = \lim_{z\to -i\sqrt{3}} \frac{\sin z}{z^2} \frac{1}{z-i\sqrt{3}} = \frac{\sin(-i\sqrt{3})}{i6\sqrt{3}}$$

Hence

$$\int_\gamma f(z)\,dz = 2\pi i\left[\frac{1}{3} - \frac{1}{6\sqrt{3}\,i}\sin(i\sqrt{3}) + \frac{1}{6\sqrt{3}\,i}\sin(-i\sqrt{3})\right]$$

$$= 2\pi i\left[\frac{1}{3} - \frac{1}{6\sqrt{3}\,i}\frac{1}{2i}\left(e^{i(i\sqrt{3})} - e^{-i(i\sqrt{3})}\right) + \frac{1}{6\sqrt{3}\,i}\frac{1}{2i}\left(e^{i(-i\sqrt{3})} - e^{-i(-i\sqrt{3})}\right)\right]$$

$$= 2\pi i\left[\frac{1}{3} + \frac{1}{12\sqrt{3}}\left(e^{-\sqrt{3}} - e^{\sqrt{3}}\right) - \frac{1}{12\sqrt{3}}\left(e^{\sqrt{3}} - e^{-\sqrt{3}}\right)\right]$$

$$= 2\pi i\left[\frac{1}{3} + \frac{1}{6\sqrt{3}}\left(e^{-\sqrt{3}} - e^{\sqrt{3}}\right)\right]$$

PROBLEM 4:

$\mathbb{C} \setminus (-\infty, 0]$ is simply connected so Riemann mapping implies that there exist a conformal (one-one, onto, analytic) map $\varphi : \mathbb{C} \setminus (-\infty, 0] \to \Delta$, where

$$\Delta = \{z \,;\, |z| < 1\}$$

Assume we can find a one-one, analytic map $\psi : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ which is onto.

Then $\psi^{-1} \circ \varphi : \mathbb{C}^* \to \Delta$.

Then $h(z) = \varphi^{-1} \circ \varphi$ is bounded so 0 is ~~a~~ removable singularity for h. Hence h extends to an entire function $\tilde{h}$ and $|\tilde{h}(z)| \leq 1$ for all $z \in \mathbb{C}$. Liouvilles theorem implies that $\tilde{h}$ is constant, since φ is one-one it follows that ψ^{-1} is constant.

~~Problem~~

PROBLEM 5:

$$\int_0^{2\pi} \frac{1}{1+\sin\theta\cos\theta}\, d\theta = \int_0^{2\pi} \frac{1}{1+\frac{1}{2}\sin 2\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{2}{2+\sin 2\theta}\, d\theta$$

$$\sin 2\theta = \frac{1}{2i}\left(e^{i2\theta} - e^{-i2\theta}\right)$$

$$= \frac{1}{2i}\left(\left(e^{i\theta}\right)^2 - \left(\frac{1}{e^{i\theta}}\right)^2\right)$$

Let $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$ or $d\theta = \frac{1}{iz}dz$

Hence

$$\int_0^{2\pi} \frac{1}{1+\sin\theta\cos\theta}\, d\theta = \int_{|z|=1} \frac{2}{2+\frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)} \frac{1}{iz}\, dz$$

Now

$$\int_{|z|=1} \frac{2}{2+\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right)} \frac{1}{iz}\, dz =$$

$$\frac{1}{i}\int_{|z|=1} \frac{2z}{2z^2+\frac{1}{2i}(z^4-1)}\, dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{4iz}{z^4+4iz^2-1}\, dz$$

$$= \int_{|z|=1} \frac{4z}{z^4+4iz^2-1}\, dz = \int_{|z|=1} f(z)\, dz$$

$$= 2\pi i \left\{\sum_{z_k} \operatorname{Res}_{z=z_k} f \;:\; z_k \in \Delta,\ f \text{ have singularity at } z_k\right\}$$

The singularities of f are at the points where

$$z^4+4iz^2-1 = 0$$

We first solve for z^2 and get

$$z^2 = \frac{-4i \pm \sqrt{-16+4}}{2}$$

$$= -2i \pm \sqrt{3}\, i$$

$$= (-2 \pm \sqrt{3})\, i$$

Notice

$|(-2-\sqrt{3})i| > 0$ so the

points where $z^2 = (-2-\sqrt{3})i$ are

not in the unit disc

If

$z^2 = (-2+\sqrt{3})i$ then

$|z^2| < 1$ so $|z| < 1$ so

the points where $z^2 = (-2+\sqrt{3})i$ are

surrounded by the unit circle

$$z^2 = (-2+\sqrt{3})i$$

$$\Downarrow$$

$$z = z_1 = ((-2+\sqrt{3})i)^{1/2}$$

or

$$z = \cancel{\#} z_2 = -((-2+\sqrt{3})i)^{1/2} = -z_1$$

$$\int_{|z|=1} \frac{4z}{z^4+4iz^2-1}\, dz = 2\pi i \left(\operatorname{Res}_{z=z_1} f + \operatorname{Res}_{z=z_2} f \right)$$

Now

$$f(z) = \frac{4z}{(z-z_1)(z-z_2)(z^2+(2+\sqrt{3})i)}$$

$$\operatorname*{Res}_{z=z_1} f(z) = \lim_{z \to z_1} \frac{4z}{(z-z_2)(z^2+(2+\sqrt{3})i)}$$

$$= \frac{4z_1}{(z_1-z_2)(z_1^2+(2+\sqrt{3})i)}$$

$$= \frac{4z_1}{2z_1((-2+\sqrt{3})i+(2+\sqrt{3})i}$$

$$= \frac{1}{\sqrt{3}\, i}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \lim_{z \to z_2} \frac{4z}{(z-z_1)(z^2+(2+\sqrt{3})i)}$$

$$= \frac{4z_2}{(z_2-z_1)(z_2^2+(2+\sqrt{3})i)}$$

$$= \frac{4z_2}{2z_2((-2+\sqrt{3})i+(2+\sqrt{3})i)}$$

$$= \frac{1}{\sqrt{3}\, i}$$

Hence

$$\int_0^{2\pi} \frac{1}{1+\sin\theta\cos\theta}\, d\theta = \int_{|z|=1} \frac{4z}{z^4+4iz^2-1}$$

$$= 2\pi i \left(\frac{1}{\sqrt{3}\, i} + \frac{1}{\sqrt{3}\, i} \right) = \underline{\frac{4\pi}{\sqrt{3}}}$$

PROBLEM 6:

a) If f is analytic in $\Delta = \{|z|=1\}$
and $f\left(\left(\frac{i}{2}\right)^n\right) = 0$ for $n = 1, 2, \ldots$,
then continuity of f implies that
$f(0) = 0$. Every disc $\Delta(0,\varepsilon) = \{|z| < \varepsilon\}$
will contain $\left(\frac{i}{2}\right)^n$ for some n so
0 is NOT an isolated zero for f,
hence f will have to be CONSTANT

b)
$u(x,y) = e^x + \cos(xy) + x^3$
is constant on the unit disc
$x^2+y^2 = |\cancel{x}+iy|^2 = |z|^2 = 1$

Let
$$h(z) = h(x+iy) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\varphi})\, P_r(\theta-\varphi)\, d\varphi$$

where $z = re^{i\theta}$
and $P_r(\theta - \varphi)$ is the Poisson kernel.

c) If v is harmonic in $\mathbb{C}$, then $v = \operatorname{Re} f(z)$ where f is analytic in $\mathbb{C}$. If $v \geq 0$ in all of $\mathbb{C}$, then

$$\operatorname{Re}(-f(z)) = -v \leq 0 \quad \text{in } \mathbb{C}$$

Let $g(z) = e^{-f(z)}$, then g is entire and

$$|g(z)| = |e^{-f(z)}| = e^{-\operatorname{Re} f(z)} \leq 1$$

Hence g is constant so v will have to be constant.