Proof. The catch here is that (i) does not prevent the two sets U^* and V^* from intersecting at points away from A. The idea of the proof is to pare U^* and V^* down to disjoint open sets, call them U and V, that continue to meet requirements (ii) and (iii). This will confirm the disconnectedness of A. To carry out this plan, write $S = U^* \sim V^*$ and $T = V^* \sim U^*$. The sets S and T are non-empty — S contains $A \cap U^*$ and T contains $A \cap V^*$ — and disjoint. Because S lies in U^* and because U^* is open, corresponding to each point z of S we can fix a number $r_z > 0$ so that the disk $\Delta(z, 2r_z)$ is contained in U^*. We then set $\Delta_z = \Delta(z, r_z)$. In like fashion we choose for each w in T a disk $\Delta_w = \Delta(w, r_w)$ in such a way that $\Delta(w, 2r_w)$ is contained in V^*. Now take $U = \bigcup_{z\in S} \Delta_z$ and $V = \bigcup_{w\in T} \Delta_w$. As unions of open sets, U and V are themselves open. Also, $A \cap U = A \cap U^* \neq \phi$, $A \cap V = A \cap V^* \neq \phi$, and, since $A \cap U^* \cap V^* = \phi$,

$$A \subset S \cup T \subset U \cup V .$$

Finally, $U \cap V = \phi$. To prove this, it suffices to show that $\Delta_z \cap \Delta_w = \phi$ whenever z belongs to S and w to T. Suppose that z_0 were an element of $\Delta_z \cap \Delta_w$ for such z and w. Assume first that $r_z \leq r_w$. In this case we would have

$$|z - w| \leq |z - z_0| + |z_0 - w| < r_z + r_w \leq 2r_w ,$$

placing the element z of S in $\Delta(w, 2r_w)$, a subset of V^*. This would clearly violate the definition of $S = U^* \sim V^*$. Similar reasoning when $r_w \leq r_z$ would lead to an inconsistency with the definition of T. Therefore $U \cap V = \phi$ and A is seen to be disconnected. ∎

3.2 Connected Sets

A plane set A is said to be *connected* if it is not disconnected. Phrased in more positive terms, the statement that A is connected asserts that the only way to include A in the union of two disjoint open sets U and V is the trivial way, which is to have A contained in either U or V. Certainly a set consisting of a single point is connected. A more significant example, one whose connectedness is closely associated with the completeness of the real number system (see Appendix A.2.2), is a line segment in $\mathbb{C}$.

Theorem 3.2. *A line segment I in the complex plane is a connected set.*

Proof. Let I have endpoints z_0 and z_1. Then I is plainly described by $I = \{(1-t)z_0 + tz_1 : 0 \leq t \leq 1\}$. Suppose that I is contained in $U \cup V$, where U and V are disjoint open sets in $\mathbb{C}$. We must demonstrate that I lies in U or that it lies in V. We shall assume that z_0 belongs to U and verify the former. (If z_0 is a point of V, then an analogous argument shows that I is contained in V.) Using the fact that $f(t) = (1-t)z_0 + tz_1$ defines a continuous function on $[0, 1]$ and that the sets U and V are open, we make the following observation: (∗) if t_0 is in $[0, 1]$ and if $f(t_0)$ is an element of U (respectively, V), then $f([0, 1] \cap [t_0 - \delta, t_0 + \delta])$ lies in U (respectively, V) for some $\delta > 0$. Consider the set J of all t in $[0, 1]$ with the property that $f([0, t])$ is a subset of U. By assumption $f(0) = z_0$ is a point of U, so $t = 0$ belongs to J. In particular, $J \neq \phi$. As a non-empty subset of $[0, 1]$, J has a supremum (= least upper bound) t_0 that belongs to $[0, 1]$. It is evident from the definition of t_0 that f must map the half-open interval $[0, t_0)$ into U. The observation (∗), coupled with the disjointness of U and V, then insures that $f(t_0)$ cannot be an element of V and so implies that $f([0, t_0])$ is a subset of U; i.e., t_0 is an element of J. If $t_0 < 1$, a second appeal to (∗) would produce a $\delta > 0$ with the feature that $f([t_0, t_0 + \delta])$ — and, hence, $f([0, t_0 + \delta])$ — is a subset of U. Thus $t_0 + \delta$ would be a member of J larger than t_0, an impossibility. The conclusions: $t_0 = 1$, $J = [0, 1]$, and $I = f(J)$ is contained in U. ∎

Taking the union of a collection of connected sets with a common point of intersection provides a mechanism for building up fairly complicated connected sets from simple ones. The principle underlying this procedure is:

Theorem 3.3. *Let $\mathcal{C}$ be a collection of connected sets in the complex plane, each of which contains a given point z_0. The union of the members of $\mathcal{C}$ is then a connected set.*

Proof. Write A for the union in question and assume that A is contained in $U \cup V$, where U and V are disjoint open sets. We must prove: A is either a subset of U or a subset of V. We suppose that z_0 belongs to U and check that A is contained in U. (The alternative case, where z_0 is an element of V, is dealt with similarly.) For this, we need only prove that each member C of $\mathcal{C}$ is a subset of U. The connectedness of any such C, together with the information that

$$z_0 \in C \subset A \subset U \cup V ,$$

allows precisely that conclusion. ∎

In combination Theorems 3.2 and 3.3 confirm that a plane set A is connected if it contains a point z_0 with the following property: for each point z of $A \sim \{z_0\}$ the line segment with endpoints z_0 and z is contained in A. (We shall describe a set of this type as *starlike with respect to* z_0.) Included in this class of sets are a number of commonplace sets that we would intuitively think of as connected and that we can now officially certify as connected according to the technical understanding of the term. Prominent among them are the complex plane itself, the open disk $\Delta(z_0, r)$, and the closed disk $\overline{\Delta}(z_0, r)$. The same two theorems just cited and a straightforward induction argument verify the connectedness of any *polygonal arc*, the name we bestow on a set $A = \bigcup_{j=1}^{n} I_j$ formed by stringing together end-to-end a finite number of line segments $I_1, I_2 \dots, I_n$, subject to the constraints that I_{j+1} be disjoint from I_k when $k < j$ and that I_{j+1} intersect I_j only at a common endpoint of these two segments. Figure 5 shows an example with $n = 6$. The points z_0 and z_1 are the *endpoints* of this polygonal arc.

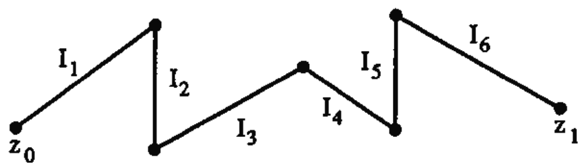

Figure 5.

3.3 Domains

A non-empty set D in the complex plane that is both open and connected is standardly referred to as a *domain* in $\mathbb{C}$. (This concept must not be confused with the "domain-set" of a function. The desire to avoid potential confusion between these two notions actually motivated our choice of the term "domain-set" — as opposed to just "domain" — for the set where a function is defined.) Throughout this book the letters D and G (for the German equivalent of "domain," *Gebiet*), when used to represent sets, will consistently stand for domains. The definition of connectedness has as a direct corollary a useful remark concerning domains.

Theorem 3.4. *If a plane domain D is expressed as the union $D = U \cup V$ of disjoint open sets U and V, then either $U = \phi$ or $V = \phi$.*

3.4 Components of Open Sets

Consider an open set U in $\mathbb{C}$ and a point z_0 of U. Let $D(z_0)$ designate the set consisting of z_0 and of all points z in U having the following property: there is a polygonal arc in U with one of its endpoints at z_0 and the other at z. Given a point z of $D(z_0)$, we set $A = \{z_0\}$ if $z = z_0$ and select a polygonal arc A in U with endpoints z_0 and z if $z \neq z_0$. We then take r small enough so that the open disk $\Delta = \Delta(z, r)$ is contained in U and, when $z \neq z_0$, intersects none of the line segments that make up A except the one with z as an endpoint. (See Figure 6.) If w is a point of Δ, then either w already lies on A — in this event w clearly belongs to $D(z_0)$ — or z_0 is linked to w by the polygonal arc in U formed when the segment from z to w is appended to A, which again places w in $D(z_0)$. In other words, Δ lies in $D(z_0)$. We have just demonstrated that $D(z_0)$ is an open set. Next,

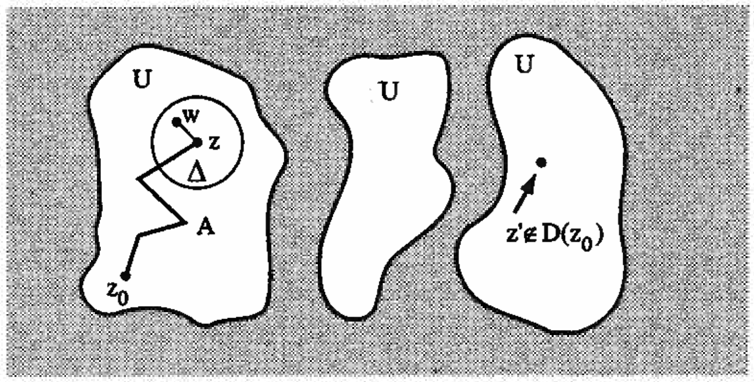

Figure 6.

if we choose for each z in $D(z_0)$, $z \neq z_0$, a polygonal arc A_z in U joining z_0 and z, it is evident that

$$D(z_0) = \bigcup_{z \neq z_0} A_z .$$

Theorem 3.3 lets us know that $D(z_0)$ is connected. Accordingly, this set is a domain. The domain $D(z_0)$ is called the *component of U containing z_0*. It is easy to see that, if z_0' is a second point of U, then $D(z_0) = D(z_0')$ when z_0' belongs to $D(z_0)$, whereas $D(z_0)$ and $D(z_0')$ are disjoint otherwise. Since every point of U belongs to some component of U, we can assert:

Theorem 3.5. *A non-empty open set U in the complex plane is the disjoint union of domains. To be specific, U is the union of its distinct components.*

For example, the open set $U = \{z : \operatorname{Re} z \neq 0\}$ has two components, the half-planes $D_1 = \{z : \operatorname{Re} z > 0\}$ and $D_2 = \{z : \operatorname{Re} z < 0\}$. The number of components of an open set may be infinite. This is true of $V = \{z : \operatorname{Re} z$ is not an integer$\}$, which has components $G_n = \{z : n < \operatorname{Re} z < n + 1\}$ for $n = 0, \pm 1, \pm 2, \cdots$. It follows from Theorem 3.4 that a plane domain D has exactly one component, itself. Therefore, $D = D(z_0)$ for every z_0 in D, which proves the next proposition.

Theorem 3.6. *Any pair of distinct points z_0 and z_1 in a plane domain D can be made the endpoints of a polygonal arc lying in D.*

Another consequence of the definition of connectedness that will be needed later is:

Theorem 3.7. *Suppose that U is an open set in the complex plane and that A is a connected subset of U. Then A is contained in some component of U.*

One of the important characteristics of continuous functions is that they preserve connectedness.

Theorem 3.8. *Let $f: A \to \mathbb{C}$ be a continuous function, and let C be a connected subset of A. Then $f(C)$ is a connected set.*

Proof. The proof is indirect. We suppose that $f(C)$ is disconnected and derive a contradiction. Assume, therefore, that U and V are open sets in $\mathbb{C}$ satisfying (i) $U \cap V = \phi$, (ii) $f(C) \cap U \neq \phi$ and $f(C) \cap V \neq \phi$, and (iii) $f(C) \subset U \cup V$. Define $C_U = \{z \in C: f(z) \in U\}$ and $C_V = \{z \in C: f(z) \in V\}$. Then (i) implies that $C_U \cap C_V = \phi$, from (ii) it follows that C_U and C_V are non-empty, and (iii) shows that $C = C_U \cup C_V$. Taking advantage of the continuity of f and the openness of U, we choose for each z in C_U an open disk Δ_z centered at z with the property that $f(A \cap \Delta_z)$ is contained in U. Similarly, we pick for each w in C_V an open disk Δ_w such that $f(A \cap \Delta_w)$ lies in V. The sets $U^* = \cup_{z \in C_U} \Delta_z$ and $V^* = \cup_{w \in C_V} \Delta_w$ are open. Furthermore, by construction $C \cap U^* = C_U$ and $C \cap V^* = C_V$. This means that: (i) $C \cap U^* \cap V^* = C_U \cap C_V = \phi$; (ii) $C \cap U^* = C_U \neq \phi$ and $C \cap V^* = C_V \neq \phi$; (iii) $C = C_U \cup C_V \subset U^* \cup V^*$. Lemma 3.1 informs us that, contrary to hypothesis, C is disconnected. This is the contradiction we sought. Thus, $f(C)$ must be connected after all. ■

4 Compact Sets

4.1 Bounded Sets and Sequences

A subset A of $\mathbb{C}$ is ***bounded*** if there is a constant $c > 0$ such that $|z| \leq c$ holds for every z in A; i.e., if A lies in some closed disk centered at the origin. A complex sequence $\langle z_n \rangle$ is bounded if there is a constant $c > 0$ such that $|z_n| \leq c$ holds for all n, or, stated differently, if $\{z_n : n = 1, 2, \cdots\}$ is a bounded set. A convergent sequence $\langle z_n \rangle$ is necessarily bounded. To see this, suppose that $z_n \to z_0$. Choose N such that $|z_n - z_0| < 1$ for all $n \geq N$ and set $c = \max\{1 + |z_0|, |z_1|, \ldots, |z_{N-1}|\}$. Then $|z_n| \leq c$ for every n. A fundamental theorem concerning the structure of the real number system states: *every bounded sequence of real numbers has a real accumulation point.* (See Theorem 2.2 in Appendix A.) This consequence of the completeness property of $\mathbb{R}$ is generally associated with the names of Bernard Bolzano (1781-1848) and Karl Weierstrass (1815-1897). It has a natural extension to complex sequences.