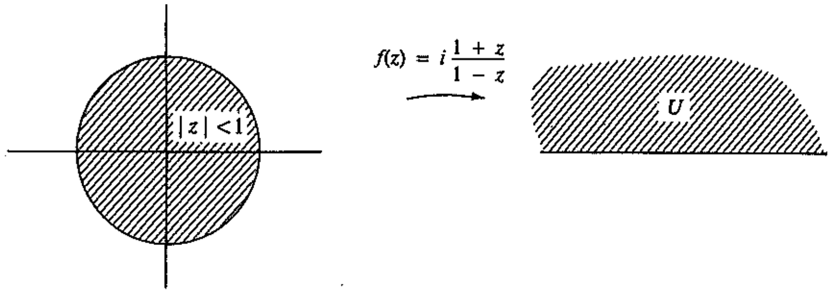


Figure 3.27

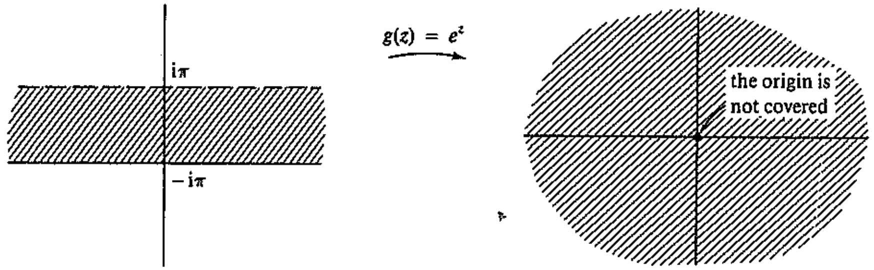


Figure 3.28

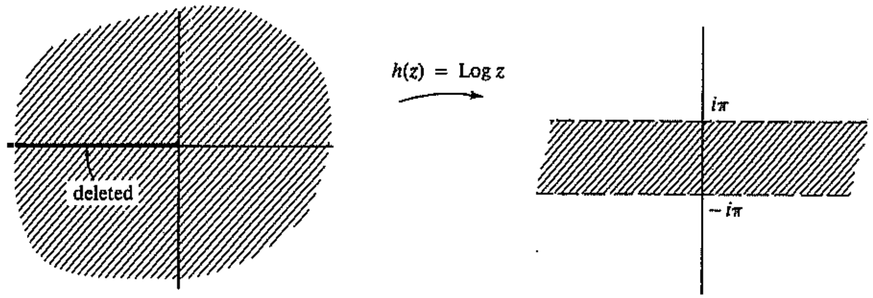


Figure 3.29

Example 3 $h(z) = \text{Log } z$ maps the region Ω obtained by deleting the ray $(-\infty, 0]$ from the plane onto the strip $\{w: |\text{Im } w| < \pi\}$ (Fig. 3.29). (See Section 5, Chapter 1.) □

Example 4 $F(z) = \sin z$ maps the strip $\{z = x + iy: 0 < x < \pi/2 \text{ and } y > 0\}$ onto the first quadrant, $\{w: \text{Re } w > 0 \text{ and Im } w > 0\}$ (Fig. 3.30). (See Section 5, Chapter 1.) □

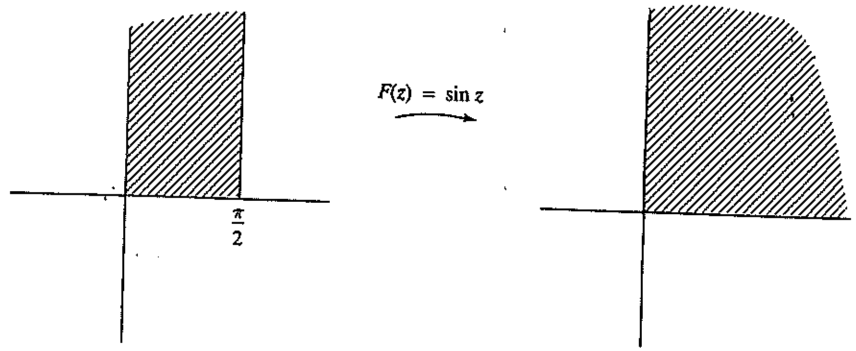


Figure 3.30

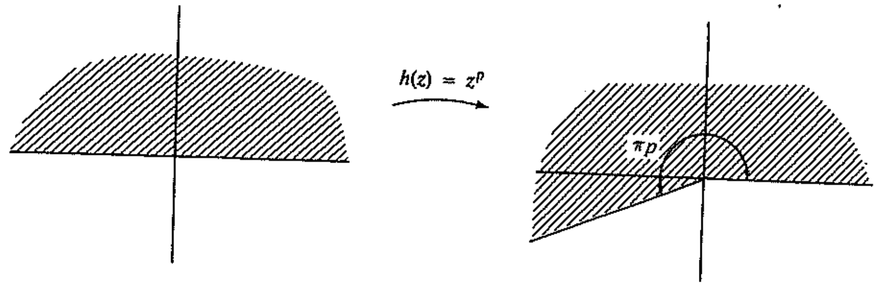


Figure 3.31

Example 5 $f(z) = z^p$, $0 < p < 2$, maps the upper half-plane U onto the region described by $\{w = re^{i\psi}: 0 < \psi < \pi p; 0 < r < \infty\}$ (Fig. 3.31). □

Schwarz*–Christoffel† Transformations From Fisher "Complex Variables"

A Schwarz–Christoffel transformation is an analytic conformal mapping of the upper half-plane onto a polygon. The key to understanding it is the examination of the behavior at the point x_0 of the function f given by

$$f(z) = A(z - x_0)^\beta + B,$$

where x_0 and β are real numbers, $0 < \beta < 2$, and A and B are complex numbers. The root is determined by choosing $\arg(z - x_0)$ to lie in the interval $(-\pi/2, 3\pi/2)$; that is, we delete from the plane the vertical ray from x_0 down.

To begin, suppose $z = x$ is real and $x > x_0$. Then $\arg f'(x) = (\beta - 1)(0) + \arg A$, so the curve parametrized by f has a tangent vector of constant slope, $\arg A$; that is, it is a straight line segment. On the other hand, if $x < x_0$, then $\arg f'(x) =$

* Hermann Amandus Schwarz, 1843–1921.

† Elwin Bruno Christoffel, 1829–1900.

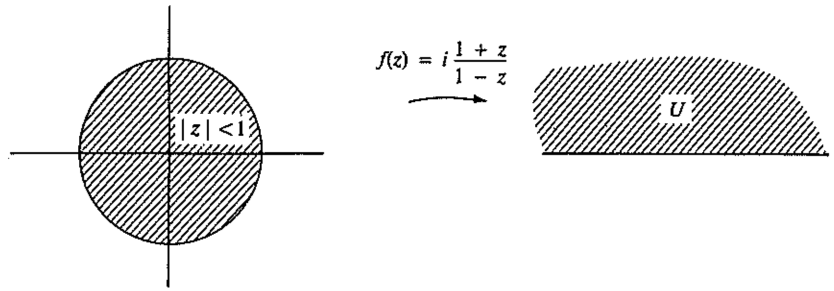


Figure 3.27

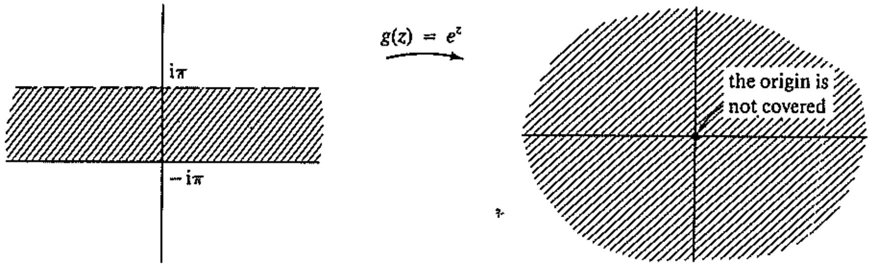
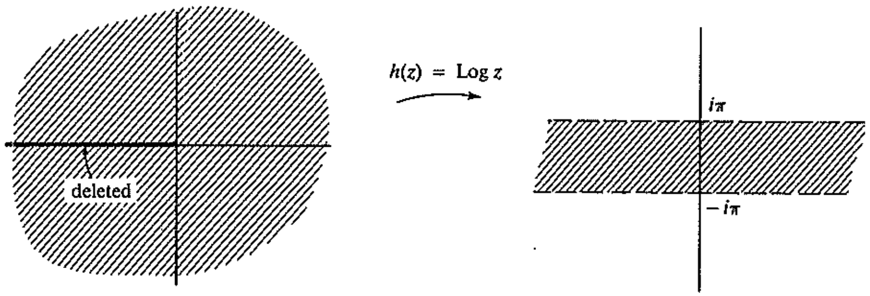


Figure 3.28



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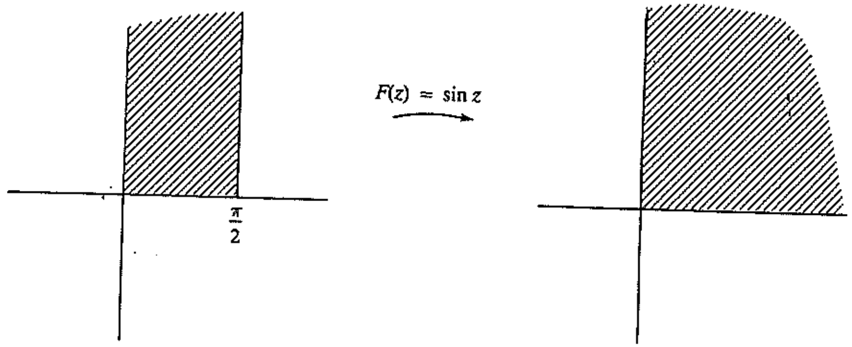

Figure 3.30

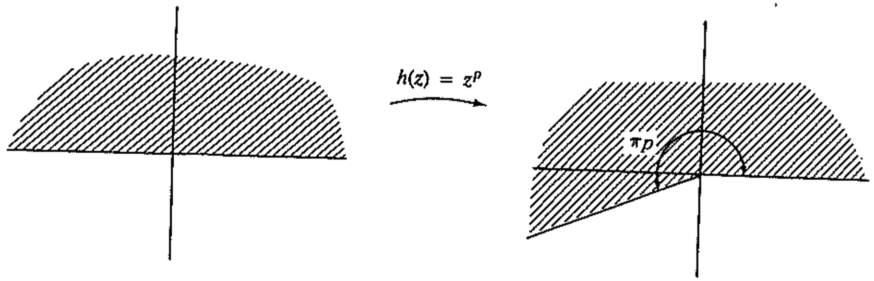

Figure 3.31

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Schwarz*-Christoffel† Transformations

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To begin, suppose $z = x$ is real and $x > x_0$. Then $\arg f'(x) = (\beta - 1)(0) + \arg A$, so the curve parametrized by f has a tangent vector of constant slope, $\arg A$; that is, it is a straight line segment. On the other hand, if $x < x_0$, then $\arg f'(x) =$

* Hermann Amandus Schwarz, 1843–1921.
† Elwin Bruno Christoffel, 1829–1900.

A full proof of Theorem 2 rests on Theorem 1 and on the reflection principle, which will be covered in Section 3 of Chapter 4. Here we shall content ourselves with several comments and a number of examples. First, we have arranged the mapping f so that $f(\infty) = w_0$, but this is not necessary; there will be times that we will want to exploit some obvious symmetries in the polygon and not require that $f(\infty) = w_0$. In general, we can select any three of the vertices w_j and any three points x_j on the real line or at ∞ and require that $f(x_j) = w_j$ for these three values of j. For example, if we demand that $f(-1) = w_2$, $f(0) = w_5$, and $f(2) = w_6$, then we have $x_2 = -1$, $x_5 = 0$, and $x_6 = 2$, so that $x_1, x_3, x_4, x_7, \ldots,$ necessarily satisfy

$$x_1 < -1 < x_3 < x_4 < 0 < 2 < x_7 < \cdots.$$

Second, Theorem 2 is stated for bounded polygons, but it holds as well for unbounded polygons—which will be the most useful cases. This can be seen most easily by looking again at the behavior of the function whose derivative is given in (1). Another technique is to obtain the unbounded polygon as a limit of bounded ones. This is sometimes useful to determine the angles $\alpha_1, \ldots, \alpha_N$ and will be illustrated in several of the examples.

Example 6 Find the Schwarz–Christoffel transformation of the upper half-plane U onto the equilateral triangle shown in Figure 3.35.

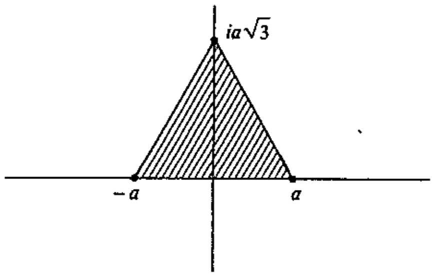

Figure 3.35

Solution The exterior angles of an equilateral triangle are all equal to $\frac{2}{3}\pi$, so that

$$\alpha_j = -\frac{\theta_j}{\pi} = -\frac{2}{3}, \qquad j = 0, 1, 2.$$

We select $x_1 = -1$ and $x_2 = 1$. Then,

$$f'(z) = A(z+1)^{-2/3}(z-1)^{-2/3}$$
$$= A(z^2-1)^{-2/3}.$$

Hence,

$$f(z) = A\int_1^z \frac{dw}{(w^2-1)^{2/3}} + B,$$

where we have selected 1 as the initial point for the integration. (Another choice would produce another B.) To find A and B, we note that

$$a = f(1) = B$$

and

$$i\sqrt{3}a = A\int_1^\infty \frac{dt}{(t^2-1)^{2/3}} + B.$$

If we denote by β the value of the integral

$$\beta = \int_1^\infty \frac{dt}{(t^2-1)^{2/3}},$$

then we find that

$$A = \frac{a(i\sqrt{3}-1)}{\beta}$$

and

$$B = a.$$

□

Example 7 Find the Schwarz–Christoffel transformation of the upper half-plane U onto the region shown in Figure 3.36.

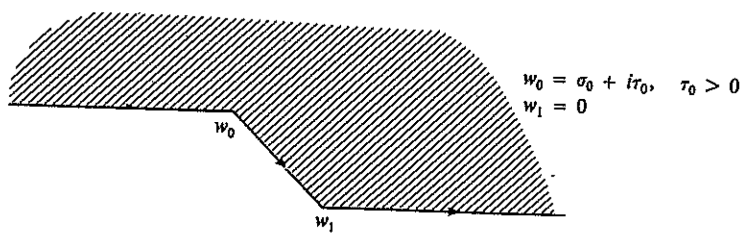

Figure 3.36

Solution The exterior angle at w_0 is θ where

$$\theta = \arctan\left(\frac{\tau_0}{\sigma_0}\right) \in (-\pi, 0),$$

7. Construct a meromorphic function on the complex plane whose poles are double poles at the points $\log n$, $n \geq 1$, with principal parts $1/(z-\log n)^2 + 1/(z-\log n)$.

8. Construct a meromorphic function on the open unit disk $\mathbb{D}$ whose poles are simple poles at the points $(1-2^{-n})e^{2\pi ik/n}$, $1 \leq k \leq n$, $n \geq 1$, with residue 1.

9. Show that $\sum_{-\infty}^{\infty} 1/(z^3 - n^3)$ converges normally to a meromorphic function. Locate the poles and find the corresponding principal parts of the function. Express the function in terms of trigonometric functions (specifically, the cotangent function).

10. Show that the lattice points $m\omega_1 + n\omega_2$, $-\infty < m,n < \infty$, can be arranged in a sequence $\{z_k\}_{k=0}^{\infty}$ such that $|z_k| \geq c\sqrt{k}$.

11. Let $\{z_k\}$ be a sequence of distinct points such that $|z_k| > c\sqrt{k}$. Show that

$$\sum \left[\frac{1}{(z-z_k)^2} - \frac{1}{z_k^2} \right]$$

converges normally on $\mathbb{C}$ and absolutely at each $z \in \mathbb{C}$.

12. Let $f(z)$ be a doubly periodic meromorphic function on $\mathbb{C}$ with periods ω_1 and ω_2, and let $\mathcal{P}(z)$ be the Weierstrass P-function associated with the periods ω_1 and ω_2. (a) Show that if the only poles of $f(z)$ are double poles at the lattice points $m\omega_1 + n\omega_2$, $-\infty < m,n < \infty$, then there are constants a and b such that $f(z) = a\mathcal{P}(z) + b$. (b) Show that if the only poles of $f(z)$ are triple poles at the lattice points $m\omega_1 + n\omega_2$, $-\infty < m,n < \infty$, then there are constants a, b, c such that $f(z) = a\mathcal{P}(z) + b\mathcal{P}'(z) + c$. (c) Show that $\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 + a\mathcal{P}(z)^2 + b\mathcal{P}(z) + c$ for some constants a, b, c.

13. Let D be a domain in $\mathbb{C}$, and let $E_k = \{|z-z_k| \leq r_k\}$, $k \geq 1$, be disjoint closed disks in D that accumulate only on the boundary of D. Suppose $Q_k(z)$ is analytic for $|z-z_k| > r_k$. Show that there is an analytic function $f(z)$ on $D \setminus \cup_{k=1}^{\infty} E_k$ such that for each k, $f(z) - Q_k(z)$ extends analytically to E_k.

From Gamelin "Complex Analysis"

3. Infinite Products

An **infinite product** is an expression of the form $\prod_{j=1}^{\infty} p_j$, where the p_j's are complex numbers. We say that the infinite product **converges** if $p_j \to 1$ and $\sum \operatorname{Log} p_j$ converges, where we sum only over terms for which $p_j \neq 0$. If the infinite product converges, we define its value to be 0 if one

of the p_j's is 0; otherwise, we define it to be

$$\prod_{j=1}^{\infty} p_j = \exp\left(\sum_{j=1}^{\infty} \operatorname{Log} p_j\right).$$

Thus any question we might ask about infinite products can be translated to a question about infinite series by taking logarithms.

To help clarify the definition of convergent infinite product, we make several simple observations. First, if $\prod p_j$ converges, then at most finitely many of the p_j's can be 0. This is because $p_j \to 1$. Second, if $\prod p_j$ converges, then

$$\prod_{j=1}^{\infty} p_j = \lim_{m\to\infty} \prod_{j=1}^{m} p_j = \lim_{m\to\infty} p_1 p_2 \cdots p_m.$$

This is because $\exp(\operatorname{Log} p_j) = p_j$. Third, we can always factor out a finite number of terms from a convergent infinite product,

$$\prod_{j=1}^{\infty} p_j = p_1 p_2 \cdots p_N \prod_{j=N+1}^{\infty} p_j.$$

Finally, if an infinite product converges, and none of the factors are 0, then the product cannot be 0.

Example. Consider

$$\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{k}\right) = (1+1)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots.$$

Since $(1+1/(2k-1))(1-1/(2k)) = 1$, the product of the first m terms is equal to 1 if m is even, and it is equal to the last factor $1+1/m$ if m is odd. Thus the product converges to 1.

It is often convenient to write $p_j = 1 + a_j$ and to express the product as $\prod(1+a_j)$. If the product converges, then $a_j \to 0$, and only finitely many of the a_j's are equal to -1. For most purposes we can ignore the terms for which $1 + a_j = 0$ and work with the "tail" $\prod_{j=N}^{\infty}(1+a_j)$ of the infinite product, for which a_j is near 0.

If $0 < t \leq 1$, we have the estimate $t/2 \leq \log(1+t) \leq t$. From this estimate it follows that if $t_j \geq 0$, then $\sum t_j$ converges if and only if $\sum \operatorname{Log}(1+t_j)$ converges. This leads immediately to the following test for convergence of infinite products.

Theorem. *If $t_j \geq 0$, then $\prod(1+t_j)$ converges if and only if $\sum t_j$ converges.*

Example. The infinite product

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^{\alpha}}\right) = (1+1)\left(1+\frac{1}{2^{\alpha}}\right)\left(1+\frac{1}{3^{\alpha}}\right)\cdots$$

converges for $\alpha > 1$ and diverges for $\alpha \le 1$.

The infinite product $\prod(1+a_j)$ is said to **converge absolutely** if $a_j \to 0$ and $\sum \mathrm{Log}(1+a_j)$ converges absolutely, where we sum over the terms for which $a_j \ne -1$. If $\prod(1+a_j)$ converges absolutely, then $\sum \mathrm{Log}(1+a_j)$ converges, and $\prod(1+a_j)$ converges.

Since $\mathrm{Log}(1+w)$ is analytic at $w=0$ and has power series expansion $\mathrm{Log}(1+w) = w + \mathcal{O}(w^2)$, we see that $|\mathrm{Log}(1+a_j)|$ is comparable to $|a_j|$ when a_j is near 0. Consequently, $\sum|\mathrm{Log}(1+a_j)|$ converges if and only if $\sum|a_j|$ converges. This together with the preceding theorem yield the following.

Theorem. *The infinite product $\prod(1+a_j)$ converges absolutely if and only if $\sum a_j$ converges absolutely. This occurs if and only if $\prod(1+|a_j|)$ converges.*

Example. We have seen that $\prod(1+(-1)^{k+1}/k)$ converges. However, it does not converge absolutely, on account of the divergence of the harmonic series, $\sum 1/k = \infty$.

Example. Consider the infinite product $\prod(1+i/k)$. Since $\mathrm{Log}(1+i/k) = i/k+\mathcal{O}(1/k^2)$, the series $\sum \mathrm{Log}(1+i/k)$ does not converge, by comparison with the harmonic series. Consequently, $\prod(1+i/k)$ does not converge. However, since $0 < \mathrm{Log}\,|1+i/k| = \frac{1}{2}\log(1+1/k^2) < 1/k^2$, the infinite product $\prod|1+i/k|$ *does* converge. Thus absolute convergence of an infinite product is not equivalent to the convergence of the product of the absolute values of the factors.

Now we turn to infinite products of functions. The Weierstrass M-test for a sum of functions is converted easily to the following test for an infinite product of functions.

Theorem. *Suppose that $g_k(x) = 1 + h_k(x)$, $k \ge 1$, are functions on a set E. Suppose that there are constants $M_k > 0$ such that $\sum M_k < \infty$, and $|h_k(x)| \le M_k$ for $x \in E$. Then $\prod_{k=1}^{m} g_k(x)$ converges to $\prod_{k=1}^{\infty} g_k(x)$ uniformly on E as $m \to \infty$.*

Choose a constant C such that $|\log(1+w)| \le C|w|$ for $|w| \le \frac{1}{2}$, and choose N such that $M_k \le \frac{1}{2}$ for $k \ge N$. The condition $|h_k(x)| \le M_k \le \frac{1}{2}$ implies that $|\mathrm{Log}(1+h_k(x))| \le CM_k$. By the Weierstrass M-test, the series $\sum_{k=N}^{\infty}\mathrm{Log}(1+h_k(x))$ converges uniformly on E. If we exponentiate we obtain uniform convergence of the partial products $\prod_{k=N}^{m} g_k(x)$ to $\prod_{k=N}^{\infty} g_k(x)$ as $m \to \infty$. Since each $g_k(x)$ is bounded, when we multiply by the first $N-1$ factors we obtain uniform convergence of $\prod_{k=1}^{m} g_k(x)$ to $\prod_{k=1}^{\infty} g_k(x)$ as $m \to \infty$.

If $G(z) = g_1(z)\cdots g_m(z)$ is a finite product of analytic functions, then by taking logarithms and differentiating, we obtain

$$\frac{G'(z)}{G(z)} = \frac{g_1'(z)}{g_1(z)} + \cdots + \frac{g_m'(z)}{g_m(z)}. \tag{3.1}$$

This procedure is called **logarithmic differentiation**. The logarithmic differentiation formula also holds for uniformly convergent infinite products of analytic functions. It is proved by applying the formula above to finite subproducts and passing to the limit.

Theorem. *Let $g_k(z)$, $k \ge 1$, be analytic functions on a domain D such that $\prod_{k=1}^{m} g_k(z)$ converges normally on D to $G(z) = \prod_{k=1}^{\infty} g_k(z)$. Then*

$$\frac{G'(z)}{G(z)} = \sum_{k=1}^{\infty}\frac{g_k'(z)}{g_k(z)}, \qquad z \in D, \tag{3.2}$$

where the sum converges normally on D.

Note that the function $G'(z)/G(z)$ has poles at the zeros of $G(z)$. However, the hypothesis implies that $g_k(z) \to 1$ uniformly on any compact subset of D, so the summands $g_k'(z)/g_k(z)$ are analytic on the compact subset for k large. Since the uniform convergence of a series is not affected by the first terms of the series, the poles do not affect the uniform convergence. Since normal convergence of a sequence of analytic functions implies uniform convergence of the derivatives of the functions in the sequence, we may apply (3.1) to the partial product $G_m(z) = g_1(z)\cdots g_m(z)$ and pass to the limit, to obtain (3.2).

Example. Consider the infinite product $f(z) = z\prod_{k=1}^{\infty}(1-z^2/k^2)$. By the Weierstrass M-test, the series $\sum|z|^2/k^2$ converges uniformly on any bounded set. Thus the infinite product converges uniformly on any bounded set, and $f(z)$ is an entire function. Since the zeros of $f(z)$ are simple zeros at the integers, we suspect that $f(z)$ is related to $\sin(\pi z)$. To check this out, we differentiate logarithmically, to obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + 2z\sum_{k=1}^{\infty}\frac{1}{z^2-k^2} = \frac{1}{z} + 2z\sum_{k=1}^{\infty}\left[\frac{1}{z-k}+\frac{1}{z+k}\right].$$

This expression we recognize from Section 2 as the partial fractions decomposition of $\pi\cot(\pi z) = \pi\cos(\pi z)/\sin(\pi z)$. We integrate, and we see that $\log f(z)$ is $\log\sin(\pi z)$ up to adding a constant. Hence $f(z) = C\sin(\pi z)$. Since $f(z)/z \to 1$ as $z \to 0$, we have $C = 1/\pi$. Thus we obtain an infinite product expansion for the sine function,

$$\sin(\pi z) \;=\; \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right) \;=\; \pi z\,(1-z^2)\left(1-\frac{z^2}{4}\right)\cdots \tag{3.3}$$

Exercises for XIII.3

1. Evaluate the following.

   (a) $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n(n+2)}\right)$ (b) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ (c) $\displaystyle\prod_{n=3}^{\infty}\left(\frac{n^2-1}{n^2-4}\right)$

2. Define $a_k = -\dfrac{1}{\sqrt{k}}$ if k is odd, and $a_k = \dfrac{1}{\sqrt{k}}+\dfrac{1}{k}+\dfrac{1}{k\sqrt{k}}$ if k is even. Show that $\prod(1+a_k)$ converges, while $\sum a_k$ and $\sum a_k^2$ diverge.

3. Show that if $t_j \geq 0$, then $\prod(1+t_j) \leq \exp(\sum t_j)$.

4. Show that if $0 < t_j < 1$, then $\prod(1-t_j)$ converges if and only if $\sum t_j$ converges.

5. Show that the infinite product $\prod(1+a_j)$ converges if and only if there is $N \geq 1$ such that $\lim_{m\to\infty}\prod_{j=N}^{m}(1+a_j)$ exists and is nonzero.

6. Show that $\prod(1+a_j)$ converges if and only if $\prod_{j=m}^{n}(1+a_j) \to 1$ as $m,n \to \infty$. *Hint.* Take logarithms and invoke the Cauchy criterion for series.

7. Show that if $\prod(1+a_k)$ converges, then $\prod|1+a_k|$ converges.

8. Suppose $a_k \to 0$. Show that the series $\sum a_k$ converges absolutely if and only if both the series $\sum \operatorname{Arg}(1+a_k)$ and $\sum \operatorname{Log}|1+a_k|$ converge absolutely.

9. Show that $\displaystyle\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \frac{e^{\pi}-e^{-\pi}}{2\pi}$.

10. Show that $\displaystyle\prod_{n=0}^{\infty}\left(1+z^{2^n}\right) = \frac{1}{1-z}$ for $|z|<1$.

11. Show that if $p_k(z)$ is a polynomial of degree k such that $p_k(0)=1$ and $p_k(z)$ has no zeros in the disk $\{|z| \leq k^3\}$, then $\prod p_k(z)$ converges normally.

12. Establish one of the following formulae, and deduce from it the other using logarithmic differentiation:

$$e^z - 1 = z e^{z/2}\prod_{k=1}^{\infty}\left(1+\frac{z^2}{4\pi^2k^2}\right),$$

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+2z\sum_{k=1}^{\infty}\frac{1}{z^2+4\pi^2k^2}.$$

13. Use the infinite product expansion for $\sin(\pi z)$ to show that the **Wallis product**

$$\prod_{k=1}^{\infty}\frac{(2k)^2}{(2k-1)(2k+1)} \;=\; \lim_{n\to\infty}\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6}{1\cdot3\cdot3\cdot5\cdot5\cdot7}\cdots\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}$$

converges to $\pi/2$. Use this to show that

$$\lim_{n\to\infty}\frac{[n!]^2}{(2n)!}\frac{2^{2n}}{\sqrt{n}} \;=\; \sqrt{\pi}.$$

14. Show that if $t>0$, then $\displaystyle\prod_{-m\leq k\leq tm}\left(1+\frac{z}{k}\right)$ converges to $\dfrac{\sin(\pi z)}{\pi z}t^z$ as $m\to\infty$.

15. Show that $\displaystyle\frac{1}{z}\prod_{n=1}^{\infty}\frac{n}{z+n}\left(\frac{n+1}{n}\right)^z$ converges to a meromorphic function $\Gamma(z)$ whose poles are simple poles at 0 and the negative integers. Show that

$$\Gamma(z) \;=\; \lim_{m\to\infty}\frac{(m-1)!\,m^z}{z(z+1)\cdots(z+m-1)}.$$

Show that $\Gamma(z+1) = z\Gamma(z)$. Show that $\Gamma(n+1)=n!$ for positive integers n. *Remark.* The function $\Gamma(z)$ is called the **gamma function.** It was first introduced by Euler, who defined it to be the limit above. We will give an equivalent definition in the next chapter.

16. Let α_k be a sequence of complex numbers, with possible repetitions, such that $|\alpha_k|<1$ and $|\alpha_k|\to 1$, and consider the **infinite Blaschke product** defined by

$$B(z) \;=\; \prod \frac{\overline{\alpha_k}}{|\alpha_k|}\frac{\alpha_k - z}{1-\overline{\alpha_k}z},$$

where the factors corresponding to $\alpha_k = 0$ are z.

(a) Suppose that $\sum(1-|\alpha_k|) < \infty$. Let E be the set of accumulation points on the unit circle $\partial\mathbb{D}$ of the α_k's. Show that the infinite product converges uniformly on compact subsets of $\mathbb{C}^*\backslash E$ to a meromorphic function $B(z)$, with the following properties: $|B(z)| < 1$ for $z \in \mathbb{D}$, $|B(z)| = 1$ for $z \in \partial\mathbb{D}\backslash E$, and $B(z)$ has zeros precisely at the points α_k.

(b) Show that if $\sum(1 - |\alpha_k|) = +\infty$, then the partial products converge uniformly on compact subsets of $\mathbb{D}$ to 0.

(c) Suppose that $f(z)$ is a bounded analytic function on $\mathbb{D}$ that is not identically zero. Show that $f(z)$ has a factorization $f(z) = B(z)g(z)$, where $B(z)$ is a (finite or infinite) Blaschke product, and $g(z)$ is a bounded analytic function on $\mathbb{D}$ with no zeros. In particular, the zeros $\alpha_1, \alpha_2, \ldots$ of $f(z)$, repeated according to multiplicity, satisfy $\sum(1 - |\alpha_k|) < +\infty$.

(The End for TMA4175)

4. The Weierstrass Product Theorem

The Weierstrass product theorem is a companion theorem to the Mittag-Leffler theorem. The Mittag-Leffler theorem asserts that we can prescribe the poles and principal parts of a meromorphic function. The Weierstrass product theorem asserts that we can prescribe the zeros and poles of a meromorphic function together with their orders.

Recall that the order of a meromorphic function $f(z)$ at a point z_0 is the order of the zero if $f(z_0) = 0$, and it is minus the order of the pole if $f(z)$ has a pole at z_0. If z_0 is neither a pole nor a zero of $f(z)$, the order of $f(z)$ at z_0 is defined to be 0.

Theorem (Weierstrass Product Theorem). *Let D be a domain in the complex plane. Let $\{z_k\}$ be a sequence of distinct points of D with no accumulation point in D, and let $\{n_k\}$ be a sequence of integers (positive or negative). Then there is a meromorphic function $f(z)$ on D whose only zeros and poles are at the points z_k, such that the order of $f(z)$ at z_k is n_k.*

The proof runs parallel to the proof of the Mittag-Leffler theorem. Let K_m be the set of $z \in D$ such that $|z| \le m$ and the distance from z to ∂D is at least $1/m$. Then K_m is a compact subset of D, $K_m \subset K_{m+1}$, and each component of $\mathbb{C}^*\backslash K_m$ contains a point of $\mathbb{C}^*\backslash D$. Suppose $z_k \in K_{m+1}\backslash K_m$. We connect z_k to a point $w_k \in C^*\backslash D$ by a simple curve γ_k in $\mathbb{C}^*\backslash K_m$. If $w_k \ne \infty$ we define $f_k(z)$ to be an analytic branch of $\log((z - z_k)/(z - w_k))$ in the simply connected domain $\mathbb{C}^*\backslash\gamma_k$. If $w_k = \infty$, we take $f_k(z)$ to be an analytic branch of $\log(1 - z/z_k)$. By Runge's theorem, there is a rational function $g_k(z)$ with only pole at w_k such that $|f_k(z) - g_k(z)| \le 2^{-k}/n_k$ on K_m. We consider the product

$$f(z) \;=\; \prod_{k=1}^{\infty}\left(\frac{z - z_k}{z - w_k}\right)^{n_k} e^{-n_k g_k(z)},$$

where we replace the factor in parentheses by $1 - z/z_k$ if $w_k = \infty$ and $z_k \ne 0$ and by z^{n_k} if both $w_k = \infty$ and $z_k = 0$. Now $n_k(f_k(z) - g_k(z))$ is not defined on γ_k. However, its exponential

$$\exp[n_k(f_k(z) - g_k(z))] \;=\; \left(\frac{z - z_k}{z - w_k}\right)^{n_k} e^{-n_k g_k(z)}$$

is meromorphic on D and has order n_k at z_k and no other poles or zeros. By the Weierstrass M-test, $\sum_{k=N}^{\infty} n_k(f_k(z) - g_k(z))$ converges uniformly on K_m, where N is chosen sufficiently large to exclude terms with $z_k \in K_m$. Hence the infinite product defining $f(z)$ converges normally on D to a meromorphic function. Clearly, $f(z)$ has the desired zeros and poles.

In the case that $D = \mathbb{C}$ is the entire complex plane, the points w_k are all ∞. Suppose $z_k \ne 0$ for $k \ge 1$, and let n_0 be the order (possibly zero) of $f(z)$ at $z = 0$. The product expansion then has the form

$$f(z) \;=\; z^{n_0}\prod_{k=1}^{\infty}\left(1 - \frac{z}{z_k}\right)^{n_k} e^{-n_k g_k(z)},$$

where $g_k(z)$ is an approximation to $\mathrm{Log}(1 - z/z_k)$ for $|z| < |z_k|$. We can take $g_k(z)$ to be a partial sum for the power series expansion of $\mathrm{Log}(1 - z/z_k)$ at $z = 0$. Since

$$\mathrm{Log}(1 - \zeta) \;=\; -\zeta - \frac{\zeta^2}{2} - \frac{\zeta^3}{3} - \cdots - \frac{z^m}{m} + \mathcal{O}\left(\zeta^{m+1}\right),$$

we can take

$$g_k(z) \;=\; -\left(\frac{z}{z_k} + \frac{z^2}{2z_k^2} + \frac{z^3}{3z_k^3} + \cdots + \frac{z^{m_k}}{m_k z_k^{m_k}}\right),$$

where m_k is chosen large enough to guarantee convergence of the product.

Exercise. Find an entire function with simple zeros at the negative integers and no other zeros.

Solution. In this case the product

$$\prod_{k=1}^{\infty}\left(1 + \frac{z}{k}\right)$$

does not converge. However, from the estimate

$$\left|\mathrm{Log}\left(1 + \frac{z}{k}\right) - \frac{z}{k}\right| \;\le\; C\frac{|z|^2}{k^2}, \qquad |z| \le R,\ k \ge 2R,$$

we see that

$$\prod_{k=1}^{\infty}\left(1 + \frac{z}{k}\right)e^{-z/k} \tag{4.1}$$

converges normally on the entire complex plane to an entire function with the prescribed zeros.