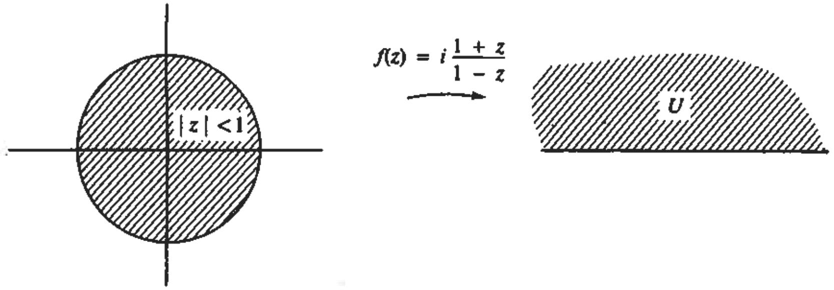

Figure 3.27

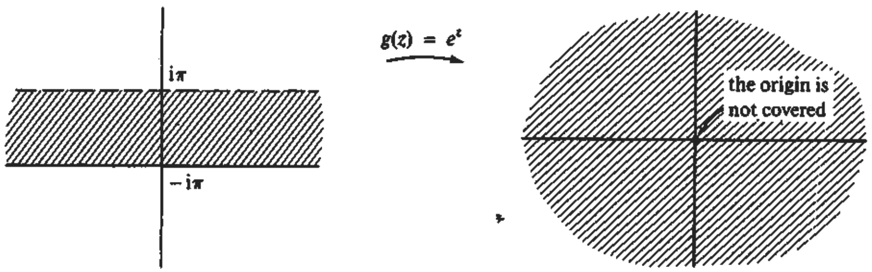

Figure 3.28

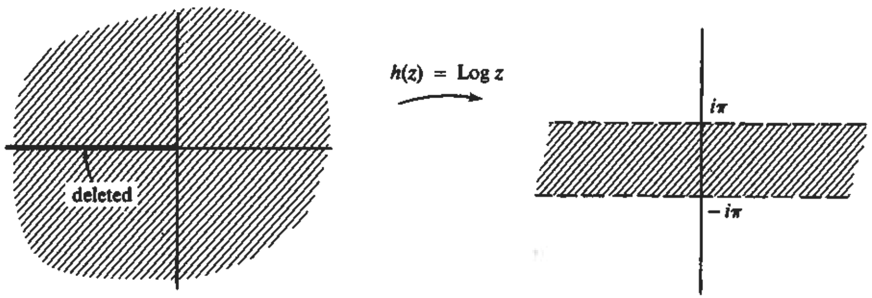

Figure 3.29

Example 3 $h(z) = \text{Log } z$ maps the region Ω obtained by deleting the ray $(-\infty, 0]$ from the plane onto the strip $\{w: |\text{Im } w| < \pi\}$ (Fig. 3.29). (See Section 5, Chapter 1.) □

Example 4 $F(z) = \sin z$ maps the strip $\{z = x + iy: 0 < x < \pi/2 \text{ and } y > 0\}$ onto the first quadrant, $\{w: \text{Re } w > 0 \text{ and Im } w > 0\}$ (Fig. 3.30). (See Section 5, Chapter 1.) □

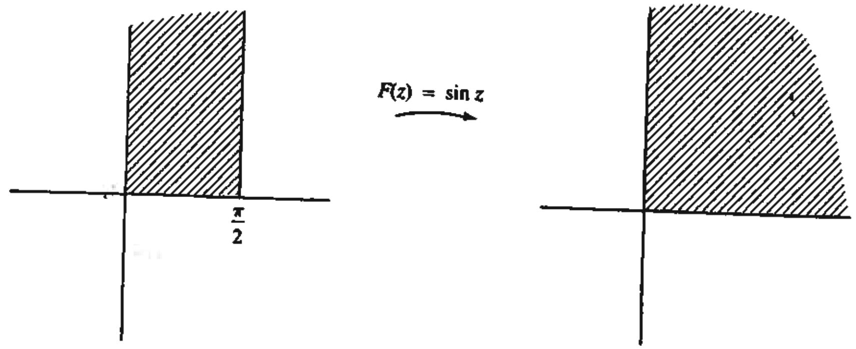

Figure 3.30

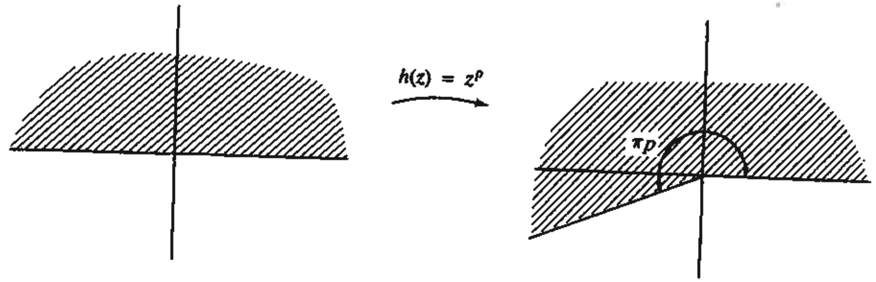

Figure 3.31

Example 5 $f(z) = z^p$, $0 < p < 2$, maps the upper half-plane U onto the region described by $\{w = re^{i\psi}: 0 < \psi < \pi p; 0 < r < \infty\}$ (Fig. 3.31). □

Schwarz*–Christoffel† Transformations

A Schwarz–Christoffel transformation is an analytic conformal mapping of the upper half-plane onto a polygon. The key to understanding it is the examination of the behavior at the point x_0 of the function f given by

$$f(z) = A(z - x_0)^\beta + B,$$

where x_0 and β are real numbers, $0 < \beta < 2$, and A and B are complex numbers. The root is determined by choosing $\arg(z - x_0)$ to lie in the interval $(-\pi/2, 3\pi/2)$; that is, we delete from the plane the vertical ray from x_0 down.

To begin, suppose $z = x$ is real and $x > x_0$. Then $\arg f'(x) = (\beta - 1)(0) + \arg A$, so the curve parametrized by f has a tangent vector of constant slope, arg A; that is, it is a straight line segment. On the other hand, if $x < x_0$, then $\arg f'(x) =$

* Hermann Amandus Schwarz, 1843–1921.
† Elwin Bruno Christoffel, 1829–1900.

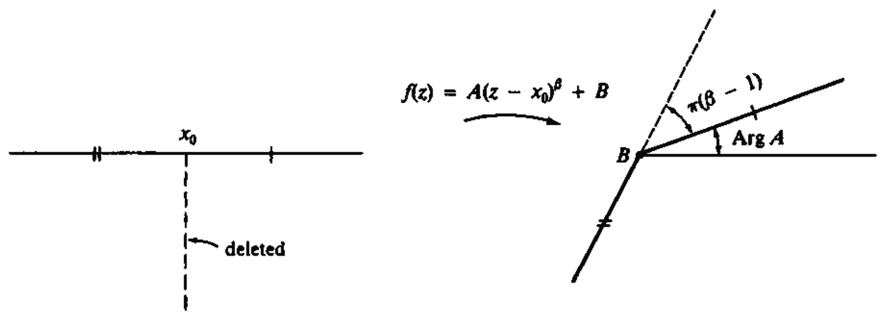

Figure 3.32

$\pi(\beta - 1) + \arg A$, so $f(x)$ lies on a straight line making an angle $\pi(\beta - 1) + \arg A$ with the positive real axis (Fig. 3.32).

Let us set $\alpha = \beta - 1$, so that $-1 < \alpha < 1$; we see that the mapping f carries the real axis into a polygonal curve of only two pieces, and the pieces meet at B with angle of $\pi\alpha = \pi(\beta - 1)$. Let us continue this idea, concentrating of f' instead of f, since it is the value of $\arg(f')$ that determines the slope of the various pieces of the image curve. Let $x_1 < \cdots < x_N$ be points on the real axis, and let $\alpha_1, \ldots, \alpha_N$ be real numbers, all in the interval $(-1, 1)$. We shall examine the behavior of the function f, whose derivative is given by

$$f'(z) = A(z - x_1)^{\alpha_1} \ldots (z - x_N)^{\alpha_N}.$$

Once again, the roots are determined by the requirement that $\arg(z - x_j)$ lie in the range $(-\pi/2, 3\pi/2)$. If x is real and near x_j, but is slightly more than x_j, then

$$\arg f'(x) = \arg A + \pi\alpha_{j+1} + \cdots + \pi\alpha_N.$$

However, if x is again real and near x_j but is slightly less than x_j, then

$$\arg f'(x) = \arg A + \pi\alpha_j + \pi\alpha_{j+1} + \cdots + \pi\alpha_N.$$

If we add to this the facts that $\arg f'(x) = \arg A$ whenever $x > x_N$ and that $\arg f'(x) = \arg A + \pi\alpha_1 + \cdots + \pi\alpha_N$ whenever $x < x_1$, we see that f maps the real axis onto a polygonal curve with $N + 1$ pieces (one or two of which may be infinite in length). See Figure 3.33.

Now we shall apply this knowledge to the mapping of the upper half-plane onto a given polygon. Let us suppose the polygon has $N + 1$ sides with vertices at points $w_0, \ldots, w_N$, arranged in the usual counterclockwise order around the polygon.

Let $\theta_0, \theta_1, \ldots, \theta_N$ be the **exterior** angles at $w_0, \ldots, w_N$, respectively (Fig. 3.34). The angles $\theta_0, \ldots, \theta_N$ lie in the range $(-\pi, \pi)$, and since the polygonal curve $\overrightarrow{w_0 w_1}$, $\overrightarrow{w_1 w_2}, \ldots, \overrightarrow{w_N w_0}$ is a simple closed positively oriented curve, we see that

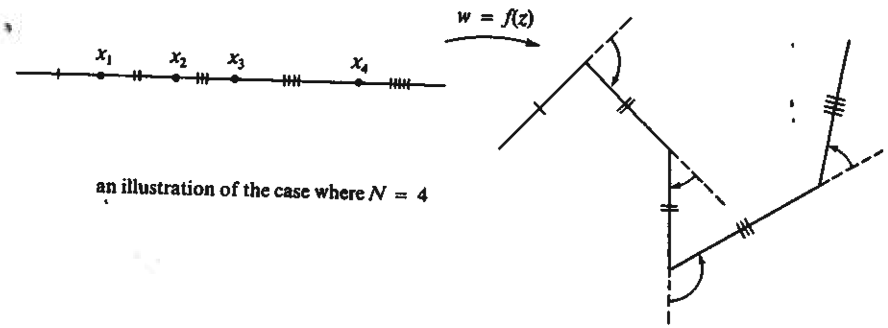

Figure 3.33 An illustration of the case where $N = 4$.

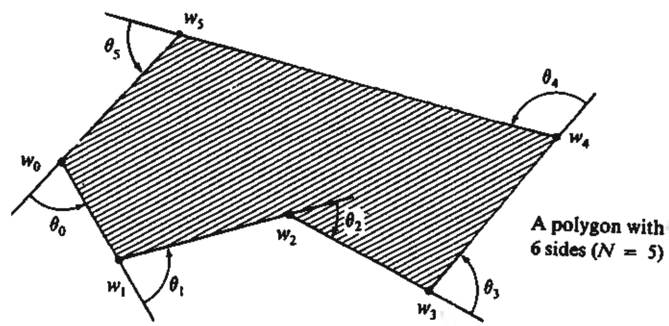

Figure 3.34 A polygon with 6 sides ($N = 5$).

$$\theta_0 + \theta_1 + \cdots + \theta_N = 2\pi.$$

Set $\alpha_j = -\theta_j/\pi$ so that

$$-1 < \alpha_j < 1 \quad \text{and} \quad \sum_{j=0}^{N} \alpha_j = -2.$$

These preliminary comments then prepare the way for the next theorem.

THEOREM 2 Schwarz–Christoffel Let P be a polygon in the plane with vertices $w_0, \ldots, w_N$ and corresponding exterior angles $\theta_0, \ldots, \theta_N$; set $\alpha_j = -\theta_j/\pi$. Then there are real numbers $x_1, \ldots, x_N$ with $x_1 < \cdots < x_N$ and a constant A such that the function $f(x)$ whose derivative is

$$f'(z) = A(z - x_1)^{\alpha_1} \ldots (z - x_N)^{\alpha_N} \tag{1}$$

gives a one-to-one analytic mapping of the upper half-plane $\operatorname{Im} z > 0$ onto the polygon P; f maps z_j to w_j for $j = 1, \ldots, N$ and $f(\infty) = \lim_{x \to \pm\infty} f(x) = w_0$. ■

A full proof of Theorem 2 rests on Theorem 1 and on the reflection principle, which will be covered in Section 3 of Chapter 4. Here we shall content ourselves with several comments and a number of examples. First, we have arranged the mapping f so that $f(\infty) = w_0$, but this is not necessary; there will be times that we will want to exploit some obvious symmetries in the polygon and not require that $f(\infty) = w_0$. In general, we can select any three of the vertices w_j and any three points x_j on the real line or at ∞ and require that $f(x_j) = w_j$ for these three values of j. For example, if we demand that $f(-1) = w_2$, $f(0) = w_5$, and $f(2) = w_6$, then we have $x_2 = -1$, $x_5 = 0$, and $x_6 = 2$, so that $x_1, x_3, x_4, x_7, \ldots$, necessarily satisfy

$$x_1 < -1 < x_3 < x_4 < 0 < 2 < x_7 < \cdots.$$

Second, Theorem 2 is stated for bounded polygons, but it holds as well for unbounded polygons—which will be the most useful cases. This can be seen most easily by looking again at the behavior of the function whose derivative is given in (1). Another technique is to obtain the unbounded polygon as a limit of bounded ones. This is sometimes useful to determine the angles $\alpha_1, \ldots, \alpha_N$ and will be illustrated in several of the examples.

Example 6 Find the Schwarz–Christoffel transformation of the upper half-plane U onto the equilateral triangle shown in Figure 3.35.

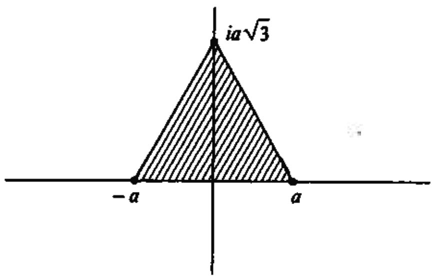

Figure 3.35

Solution The exterior angles of an equilateral triangle are all equal to $\frac{2}{3}\pi$, so that

$$\alpha_j = -\frac{\theta_j}{\pi} = -\frac{2}{3}, \qquad j = 0, 1, 2.$$

We select $x_1 = -1$ and $x_2 = 1$. Then,

$$f'(z) = A(z+1)^{-2/3}(z-1)^{-2/3}$$
$$= A(z^2 - 1)^{-2/3}.$$

Hence,

$$f(z) = A\int_1^z \frac{dw}{(w^2-1)^{2/3}} + B,$$

where we have selected 1 as the initial point for the integration. (Another choice would produce another B.) To find A and B, we note that

$$a = f(1) = B$$

and

$$i\sqrt{3}a = A\int_1^\infty \frac{dt}{(t^2-1)^{2/3}} + B.$$

If we denote by β the value of the integral

$$\beta = \int_1^\infty \frac{dt}{(t^2-1)^{2/3}},$$

then we find that

$$A = \frac{a(i\sqrt{3}-1)}{\beta}$$

and

$$B = a. \qquad \square$$

Example 7 Find the Schwarz–Christoffel transformation of the upper half-plane U onto the region shown in Figure 3.36.

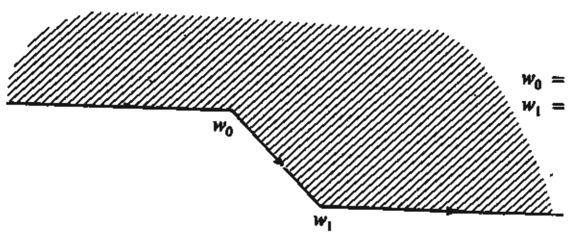

Figure 3.36

Solution The exterior angle at w_0 is θ where

$$\theta = \arctan\left(\frac{\tau_0}{\sigma_0}\right) \in (-\pi, 0),$$