

# CLASS NOTES on Riemann Mapping Theorem.

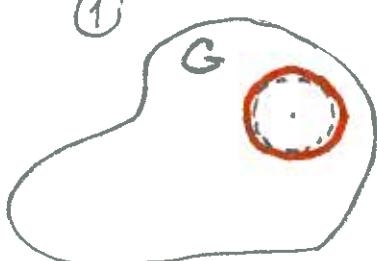
First, we may be a bit more concrete in stating "Theorem X.19" (although it does not simplify the proof very much).

**Theorem (Montel)** Suppose that  $\mathcal{F}$  is a family of holomorphic functions  $f : G \rightarrow \mathbb{D}$  where  $G$  is a domain in  $\mathbb{C}$  and  $\mathbb{D} = D(0,1)$  the open unit disk. Then every sequence  $(f_n)_{n=1}^{\infty}$  of functions belonging to  $\mathcal{F}$  has a locally uniformly convergent subsequence. (The limit function is holomorphic by Weierstrass C.T.)

Recall A sequence is "locally uniformly convergent" if each point in the domain has a neighborhood in which the sequence is uniformly convergent, or equivalently, if the sequence is uniformly convergent on each compact subset.

## The proof.

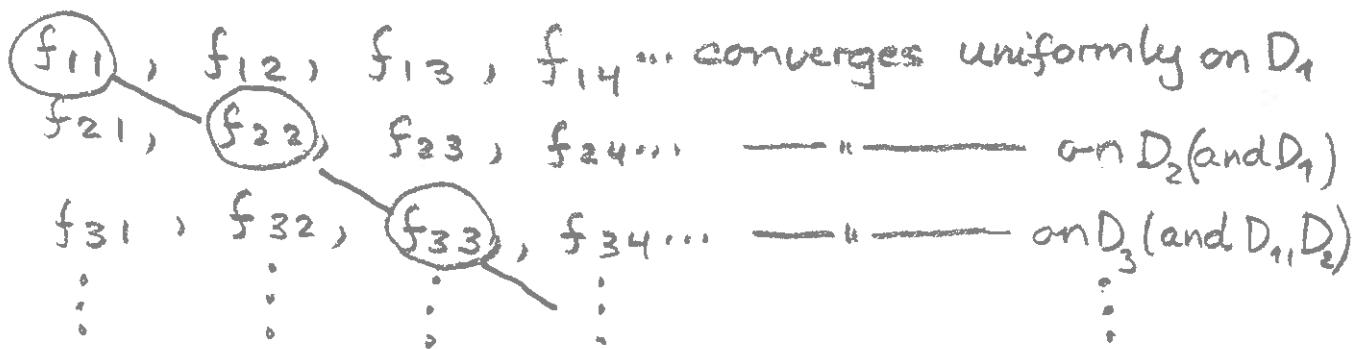
①



It suffices to prove that  $(f_n)$ , for each open disk whose closure is also in  $G$ , has a loc. unif. conv. subseq. in the disk.

Why? First observe that  $G = \bigcup_{k=1}^{\infty} D_k$  where the centers of  $D_k$ ,  $s_k$ , are the rational points in  $G$  while the radius of  $D_k$ ,  $r_k$ , satisfies  $r_k < \text{dist}(s_k, \partial G)$ . (The disks can be numbered since the rationals can be numbered). By our supposition, the sequence  $(f_n)$  has a subsequence  $f_{n_1}, f_{n_2}, f_{n_3}, \dots$ , that we denote by  $(f_{1n})_{n=1}^{\infty}$ , that converges uniformly on  $D_1$ . Next we can extract a subsequence of  $(f_{1n})_{n=1}^{\infty}$ ,  $(f_{2n})_{n=1}^{\infty}$ , so that  $(f_{2n})_{n=1}^{\infty}$  converges

uniformly on  $D_2$ . The scheme is the following



The diagonal sequence  $g_n = f_{nn}$  ( $n = 1, 2, 3, \dots$ ) converges locally uniformly in  $G$ .

② Now consider one of our disks  $D_k$ . Since  $\overline{D}_k$  is contained in  $G$  there is an open disk in  $G$  containing  $\overline{D}_k$ . A linear change of variables maps the larger disk (stapled on figure) onto the unit disk  $\mathbb{D}$ .



To prove the assertion that  $f_n: \mathbb{D} \rightarrow \mathbb{D}$  has a subsequence that converges locally uniformly, we assume that  $f_n$  has the power series  $\sum a_{n,k} z^k$ . Here

$$|a_{n,k}| = \left| \frac{1}{2\pi i} \int_{C_r} \frac{f_n(z)}{z^{k+1}} dz \right| \leq \frac{1}{2\pi} \frac{2\pi r}{r^{k+1}} = \frac{1}{r^{k+1}}$$

where  $0 < r < 1$ . We see that  $|a_{n,k}| \leq 1$  when we let  $r \rightarrow 1$ . Each sequence  $(a_{n,k})_{n=1}^{\infty}$  ( $k=0, 1, 2, \dots$ ) is bounded and so has a convergent subsequence. Now, using a diagonalization procedure in the same way as above, we can obtain a sequence  $(a_{n_j, k})_{j=1}^{\infty}$  that converges for each  $k$  (Ex. 8.19.1; Owing 11)

Let  $a_k = \lim_{j \rightarrow \infty} a_{n_j, k}$  ( $k = 0, 1, 2, \dots$ ). We have

$|a_k| \leq 1$  for all  $k$ , so the power series

$\sum_{k=0}^{\infty} a_k z^k$  converges in  $\mathbb{D}$  ( $R = \frac{1}{\limsup |a_k|^{\frac{1}{k}}}$ ),

say to the function  $f$ . It remains to show that the sequence  $f_{n_j} \rightarrow f$  locally uniformly in the disk.

Fix  $r \in (0, 1)$ . It is enough to show that  $f_{n_j}(z) \rightarrow f(z)$  uniformly for  $|z| \leq r$ . Clearly

$$\begin{aligned} |f(z) - f_{n_j}(z)| &= \left| \sum_{k=0}^{\infty} (a_k - a_{n_j, k}) z^k \right| \\ &\leq \sum_{k=0}^m |a_k - a_{n_j, k}| + 2 \sum_{k=m+1}^{\infty} r^k \\ &= \sum_{k=0}^m |a_k - a_{n_j, k}| + \frac{2r^{m+1}}{1-r}. \end{aligned}$$

So  $\epsilon > 0$  given,

we choose  $m$  so large that  $\frac{2r^{m+1}}{1-r} < \epsilon/2$  and next  $j_0$  so large that  $|a_k - a_{n_j, k}| < \epsilon/2(m+1)$  for  $k = 0, \dots, m$  whenever  $j \geq j_0$ . Then  
 $|f(z) - f_{n_j}(z)| < \epsilon$  for  $|z| \leq r$ , the desired conclusion

### Riemann Mapping Theorem

If  $G$  is a simply connected domain different from  $\mathbb{C}$ , then there exists a univalent holomorphic map of  $G$  onto the open unit disk  $\mathbb{D}$ .

The proof.

Fix  $z_0 \in G$  and let  $\mathcal{F} = \{f: G \xrightarrow[\text{holo}]{} \mathbb{D} \mid f(z_0) = 0\}$

**I**  $\mathcal{F} \neq \emptyset$ : For this it is enough to show that there exists a bounded univalent holomorphic function in  $G$ , since we then can compose

with a Möbiustransformation and get a function in  $\mathbb{F}$ .

Let  $c \in \mathbb{C} \setminus G$  and look at a branch of  $\log(z - c)$ ,  $L$  say. (This branch exists since  $G$  is simply connected.) The holo function  $L$  is univalent:  $L(z_1) = L(z_2) \Rightarrow z_1 - c = z_2 - c$ . Also  $L(G)$  is an open set. Moreover,  $L(G) \cap \{L(G) + 2\pi i\} = \emptyset$ . (Suppose  $L(z_1) = L(z_2) + 2\pi i$ . Then  $z_1 - c = e^{2\pi i}(z_2 - c) = z_2 - c$ )

Have

$L(G) + 2\pi i$

$$|L(z_0) - (L + 2\pi i)| \geq \text{dist}(L(z_0), \partial L(G)) = \varepsilon > 0$$

$L(G) \setminus \{L(z_0)\}$

$$\Leftrightarrow \left| \frac{1}{L - L(z_0) - 2\pi i} \right| \leq \frac{1}{\varepsilon}; (L - L(z_0) - 2\pi i)^{-1} \text{ bounded.}$$

II Let  $\varrho = \sup \{ |f'(z_0)| : f \in \mathbb{F} \} > 0$ .

Let  $(f_n)_{n=1}^{\infty}$  be a sequence from  $\mathbb{F}$  s.t.  $|f_n'(z_0)| \rightarrow \varrho$

By Montel (or Stieltjes-Osgood if you like) this sequence has a locally uniformly convergent subsequence. To simplify the notation, we may assume that  $(f_n)$  converges loc. uniformly. Let  $f$  denote the limit function. By Weierstrass C.T.  $f$  is holomorphic, and  $f_n' \rightarrow f'$  loc. uniformly.

$$\lim f_n'(z_0) = f'(z_0) \Rightarrow \underline{\varrho} = \lim |f_n'(z_0)| = \underline{|f'(z_0)|}$$

Hence  $f$  is non-constant. Moreover  $f$  is univalent since every  $f_n$  is univalent. (Hurwitz Theorem, Ex. X.12.5; Owing10) Clearly

$|f(z)| \leq 1, z \in G, f(z_0) = 0$ . Since  $f(D)$  is an open set,  $|f(z)| < 1, z \in G$ , and  $f \in \mathbb{F}$ . Hence  $f$  is a solution to our extremal problem.

III  $f(G) = \mathbb{D}$  : Suppose  $f(G) \subsetneq \mathbb{D}$ . Let  $a \in \mathbb{D} \setminus f(G)$  and define  $h_1$  by

$$h_1 = \frac{f-a}{1-\bar{a}f}$$



$h_1 = A \circ f$  where  $A: \mathbb{D} \rightarrow \mathbb{D}$  is a Möbius transf. mapping  $a$  to 0. Since  $a \notin f(G)$ ,  $h_1$  does not vanish on  $G$ , and since  $G$  is simply connected, there is a branch  $h_2$  of  $\sqrt{h_1}$  defined in  $G$ .  $h_2$  is also a univalent holomorphic map of  $G$  onto  $\mathbb{D}$ . Lastly we define

$$g = \frac{h_2 - b}{1 - \bar{b}h_2} \text{ where } b = h_2(z_0) \quad (|b| < 1)$$

so that  $g(z_0) = 0$ , i.e.  $g \in \mathbb{F}$ .

Some calculations give  $f = \left( \frac{g+\alpha}{1+\bar{\alpha}g} \right) g$  where  $\alpha = \frac{2b}{1+|b|^2}$  which in turn gives

$$f'(z_0) = \alpha g'(z_0) \Rightarrow |f'(z_0)| < |g'(z_0)|$$

$$\text{since } |\alpha| < 1 ! \quad \left( \frac{2|b|}{1+|b|^2} < 1 \Leftrightarrow (1-|b|^2) > 0 \right)$$

We have reached a contradiction, and so  $f(G) = \mathbb{D}$ .

THE END !

(Step III is quite technical. The details here will not be on the exam!)