Solutions to the exam in TMA4175
May 20, 2009

<u>Problem 1</u> We will use the Cauchy-Hadamard formula, and are given $R_1 = \frac{1}{\overline{\lim}\,|a_n|^{\frac{1}{n}}}$, $0 < R_1 < \infty$, and $R_2 = \frac{1}{\overline{\lim}\,|b_n|^{\frac{1}{n}}} = \infty$, i.e. $\overline{\lim}\,|a_n|^{\frac{1}{n}} = \frac{1}{R_1}$, $\overline{\lim}\,|b_n|^{\frac{1}{n}} = 0$. Since $\overline{\lim}\,|a_n b_n|^{\frac{1}{n}} \leq \overline{\lim}\,|a_n|^{1/n}\,\overline{\lim}\,|b_n|^{1/n} = 0$, the convergence radius of $\sum a_n b_n z^n$ is ∞.

<u>Problem 2</u> a) $f(z) = e^z - 1 = 0 \iff e^x e^{iy} = 1$

$\iff \underline{x = 0,\ y = k2\pi;\ k \in \mathbb{Z}}$

<u>The zeros of f are $z = 2k\pi i$, $k \in \mathbb{Z}$</u>

$f'(z) = e^z$; $e^{2k\pi i} = 1 \neq 0$.

<u>The zeros are of order 1</u>

b) The residues in $2k\pi i$ are $\frac{2k\pi i}{1} = \underline{2k\pi i}$

$$\left(\operatorname{res}_{z_0} \frac{z}{e^z - 1} = \frac{z_0}{(e^z-1)'(z_0)} = \frac{z_0}{1} \right)$$

$$\boxed{\int_\gamma \frac{z}{e^z - 1}\,dz} = 2\pi i \left[\operatorname{ind}_\gamma(-4\pi i)\cdot(-4\pi i) + \operatorname{ind}_\gamma(-2\pi i)(-2\pi i) + \operatorname{ind}_\gamma(0i)\cdot 0 + \operatorname{ind}_\gamma(2\pi i)\cdot(2\pi i) \right]$$

$$= 2\pi i\left[-4\pi i + 2(-2\pi i) + 2\pi i\right] = \boxed{12\pi^2}$$

Problem 3 We will use Rouché's Theorem to prove that all the roots of $\underbrace{2z^5-6z^2+z+1}_{p(z)}=0$ lie in the disk $|z|<2$.

First
$$|\overbrace{2z^5-6z^2+z+1}^{p(z)}-2z^5| \le 6\cdot 4+2+1<|2z^5|$$
when $|z|=2$. Hence $p(z)$ and $2z^5$ have the same number of zeros in $|z|<2$ i.e. obviously 5. Since $p(z)$ has degree 5 this means all of the zeros! (As usual we take into account multiplicities.)

Problem 4 If f is a holomorphic function in a domain D (open and connected set in $\mathbb{C}$), and p a polynomial, then $g=p\circ f$ is a real valued holomorphic function and hence a constant. This is a well known fact, and was early in the book (and in 4K?) proved using the Cauchy-Riemann. Now we can refer to g as an "open map" if g is not a constant. In any case we have $p(f(z))=c$, $z\in D$, and the possible values for f are the finitely many roots of $p(w)=c$. But since $f(D)$ is a connected set, $f(D)$ is only one point!

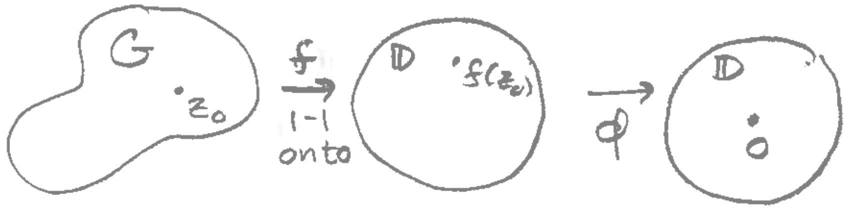
Problem 5 Since $f(z)\neq 0$ on $|z|\le 1$, we can apply the maximum modulus principle to $\frac{1}{f(z)}$. Then $\frac{1}{|f(z)|}$ has no maximum in $|z|<1$ $\Leftrightarrow$ $|f(z)|$ has no minimum in $|z|<1$. But $\frac{1}{|f(z)|}$ must have a maximum on the compact $|z| \leq 1$. So this must occur on $|z|=1$. This means that $|f(z)|$ attains its minimum on $|z|=1$.

Problem 6 If f is an entire function, so is if. Now $f = u + iv \Rightarrow if = -v + iu$. Hence $|e^{if}| = e^{-v}$, and $|v| \leq M$ implies that the entire function e^{if} is bounded, and hence a constant by Liouville. Moreover, $e^{if} = c$ $\Rightarrow |e^{if}| = e^{-v} = |c| > 0 \Rightarrow -v = \ln|c| \Rightarrow \underline{v = -\ln|c|}$. v must be a constant.

Problem 7 $f(z) = az + b$; $a, b \in \mathbb{C}, a \neq 0$ provide univalent holomorphic maps of $\mathbb{C}$ onto $\mathbb{C}$. We must prove these are the only ones.
A holomorphic function $f: \mathbb{C} \xrightarrow[\text{onto}]{1-1} \mathbb{C}$ has a singularity at ∞. Since f is 1-1 and onto, ∞ is not a removable or essential singularity. Therefore, f has a pole of some order m at ∞, i.e. the Taylor expansion of f reduces to $f(z) = a_0 + a_1 z + \cdots + a_m z^m$ where $m \geq 1$ and $a_m \neq 0$
If $m > 1$, $f'(z) = a_1 + \cdots + m z^{m-1} = 0$ would have a solution, in contradiction to f being univalent.

Problem 8

We look at



Let the new f, f_0, be $f_0 = \phi \circ f$ where

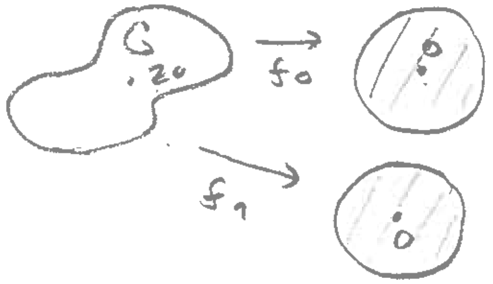
$$\phi(z) = \lambda \frac{z - f(z_0)}{\overline{f(z_0)}\, z - 1}$$ and the rotation

λ is chosen so that $f_0'(z_0) = \phi'(f(z_0)) \cdot \overset{\neq 0}{f'(z_0)} > 0$:

$$\phi'(f(z_0)) = -\frac{\lambda}{1 - |f(z_0)|^2} .$$

Let $-\lambda = e^{-i \operatorname{Arg} f'(z_0)}$

f_0 is unique, for let f_1 have the prop. $f_1(z_0) = 0$, $f_1'(z_0) > 0$



$$(f_1 \circ f_0^{-1})(0) = 0 \Rightarrow f_1 = \lambda f_0$$

$f_1 \circ f_0^{-1} = \lambda\, id$

$$f_1'(z_0) \cdot \frac{1}{f_0'(z_0)} > 0 \Rightarrow \lambda = 1$$