Lecture 1

1. Periodic functions

In the first part of the course we study Fourier series. The Fourier series is a tool to study periodic functions by writing it as an infinite sum of trignomic polynomials.

Definition 1.1. A function on $\mathbb{R}$ is called *periodic* provided $f(x+T) = f(x)$ for some $T > 0$. We usually tacitly assume T to be the smallest positive period of f.

We identify the following three things:

(a) A periodic function on $\mathbb{R}$ with period 2π.
(b) A function f on $[0, 2\pi]$ with $f(0) = f(2\pi)$.
(c) A function defined on the unit circle.

If we have a periodic function with period 2π, we can restrict the function to $[0, 2\pi]$ or any interval of length 2π, like $[-\pi, \pi]$. Conversely, if we have a function defined on $[0, T]$ satisfying $f(0) = f(T)$, then we can construct a periodic function on $\mathbb{R}$ by its periodization $f_T(x) = \sum_{k\in\mathbb{Z}} f(x - kT)$ (with modification at integral multiples of T).

The unit circle can be parametrized on $[0, 2\pi]$ by $e^{i\theta}$. Let F be a function on the circle, then $f(\theta) := F(e^{i\theta})$ is a periodic function of period 2π. Similarly, we can also identify F with $g(x) = F(e^{2\pi i x})$, then $g(x)$ is a periodic function with period 1. F is continuous, differentiable, or integrable... if and only if f the corresponding f is continuous, differentiable, or integrable. Let $\mathbb{T}$ denote $\mathbb{R}/\mathbb{Z}$, we also identify the circle as $\mathbb{T}$.

Let f be a periodic function with period T, then $f(Tx)$ is a periodic function with period 1, hence we can restrict our study on period 1 or period 2π case and the period T case can be obtained by scaling. Note that the integral of function on its period is not invarient under scaling: $\int_0^1 f(Tx)dx = \int_0^T f(x')d(\frac{x'}{T}) = \frac{1}{T}\int_0^T f(x')dx'$, i.e. shrinking period by T also shrinks integral by T.

Trignomic functions and trignomic polynomials

A real trignomic polynomial is a function of the form $f(t) = \sum_{k=1}^n (a_k \cos(kt) + b_k \sin(kt)) + \frac{a_0}{2}$. In most theoretical calculations will use the complex exponentials instead of sines and cosines, thanks to the identity $e^{i\theta} = \cos\theta + i\sin\theta$. Then $\cos t = \frac{e^{it}+e^{-it}}{2}$, $\sin t = \frac{e^{it}-e^{-it}}{2i}$, and

$$f(t) = \sum_{k=1}^n a_k \frac{e^{it}+e^{-it}}{2} + \sum_{k=1}^n b_k \frac{e^{-it}-e^{-it}}{2i} + \frac{a_0}{2}$$

$$= \sum_{k=1}^n e^{ikt}\frac{a_k - ib_k}{2} + \sum_{k=1}^n e^{-ikt}\frac{a_k+ib_k}{2} + \frac{a_0}{2}.$$

Let $c_k = \frac{a_k - ib_k}{2}$, in particular $c_0 = \frac{a_0+0}{2}$, we have $f(t) = \sum_{k=-n}^n c_k e^{ikt}$ with $c_{-k} = \overline{c_k}$. Note that this is because our $f(t)$ is real at the beginning.

Example 1.1. $D_N := \sum_{n=-N}^N e^{int}$ is a trignomic polynomial and $D_N = \frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{1}{2}t)}$.

Proof. Let $\omega = e^{it}$,then $D_N = \sum_{n=-N}^N \omega^n = \sum_{n=0}^N \omega^n + \sum_{n=-N}^{-1}\omega^n$. The first term = $\frac{1-\omega^{N+1}}{1-\omega}$, the second term = $\frac{\frac{1}{\omega}(1-\frac{1}{\omega^N})}{1-\frac{1}{\omega}} = \frac{\omega^{-N}-1}{1-\omega}$.

Then sum up we get $D_N = \frac{\omega^{-N}-\omega^{N+1}}{1-\omega} = \frac{\omega^{-(N+1/2)}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}} = \frac{2i\sin((N+\frac{1}{2})x)}{2i\sin(\frac{1}{2}x)}$.

Question: Suppose I tell you that a function $f(t)$, for example, $\frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{1}{2}t)}$, is a trignomic polynomial, how do you know its coefficients?

To answer this question we need the following fundamental property of complex exponentials.

Lemma 1.1

$$\int_0^{2\pi} e^{int}e^{-imt} = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases} \tag{1}$$

The vector space of trignomic polynomials

Let V be a vector space with inner product $(,)$, let $e_1, \dots, e_m$ be an orthonormal basis of V. Then every vector $\mathbf{x}$ in V can be written as $a_1e_1 + \cdots + a_ne_n$ for scalers $a_1, \dots, a_n$, and we can calculate a_k by $a_k = (v, e_k)$. In fact, we have $(v, e_k) = (a_1e_1 + \cdots + a_ne_n, e_k) = 0 + a_k(e_k, e_k) = a_k$.

Now let V_N be trignomic polynomials of degree N, i.e, V_N consists of functions of the form $\sum_{n=-N}^{N} c_ne^{int}, c_n \in \mathbb{C}$. Then V_N is a complex vector space. Define inner product $(,)$ on V_N by $(f, g) \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt$, then $(e^{ikt})_{k=-N}^{N}$ is an orthonormal basis of V_N. So we have $c_n = (f, e_n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{e^{int}}dt = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$ and $f = \sum_{n=-N}^{N} c_ne^{int}$.

In the period 1 case the inner product is $\int_0^1 f(x)\overline{g(x)}dx$ and $(e^{2\pi inx})_{n=-N}^{N}$ is orthonormal basis.

2. Fourier series

Can we write any period function, say, with period 2π, as a sum of complex exponentials? One can soon realize that finite sum will usually not be possible because a trignomic polynomial is always smooth. So we have to consider infinite sums.

Definition 2.1. Let f be an integrable function on $[0, T]$.

- The *n-th Fourier coefficient* of f is given by $c_n = \frac{1}{T}\int_0^T f(t)e^{-(2\pi/T)int}dt$. Sometimes we also use $\hat{f}(n)$ to denote c_n.
- $S_N(f) := \sum_{n=-N}^{N} c_ne^{(2\pi/T)int}$ is the N-th Fourier partial sum of f.
- The series $S(f) := \sum_{n=-\infty}^{+\infty} c_ne^{(2\pi/T)int} = \lim_{N\to\infty} S_N(f)$ is called the Fourier series of f.

The first question is convergence of Fourier series, when and in which sense can we write $f(x) = \sum_{n=-\infty}^{+\infty} c_ne^{inx}$?

Note. In order for the integrals to make sense we need some integrability assumption on f. Now you can understand the notion "integrable" as "Riemann integrable". Later, we'll introduce a more general notion of *Lebesgue integrals*, most of the results holds for Lebesgue integrable functions.

The convergence of Fourier series turns out to be delicate.

- If the function is continuously differentiable, then the Fourier series converges uniformly.
- There exists a continuous function whose Fourier series diverge at some point.
- In the general case, need to consider more general sense of convergence.

Example 2.1 $f(\theta) = \theta$ on $[-\pi, \pi]$. Then the Fourier series of f is given by

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta d(\frac{e^{-in\theta}}{-in}) = \frac{1}{2\pi}(\theta e^{-in\theta}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{-in}d\theta) = \frac{(-1)^{n+1}}{in}$

when $n \neq 0$ and $c_0 = 0$. So $S(f) = \sum_{n=-\infty}^{+\infty} \frac{(-1)^{n+1}}{in}e^{in\theta} = 2\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\sin n\theta}{n}$.

We can use the following Sagemath code to visualize the Fourier partial sums, you can change N and see how it goes. This approach is to directly plot the Fourier partial sum based on our hand calculation.

```
x = var('x') # Define symbolic variable x
n = var('n') # Define symbolic variable n
N = 20
a(n) = 2*(-1)^(n+1)*(1/n)
Sf=sum(a(n)*sin(n*x),n,1,N) # The Fourier partial sum of f
plot(Sf,x,-pi,pi)
```

Example 2.2 $f(x) \equiv 1$ on $[-1/2, 1/2]$, find the Fourier series of f on $[-T/2, T/2]$.

$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot e^{-(2\pi/T)inx}dx = -\frac{1}{2\pi in}(e^{-\frac{n\pi}{T}i} - e^{-\frac{n\pi}{T}i}) = \frac{\sin(\frac{n\pi}{T})}{n\pi}$ for $n \neq 0$ and $c_0 = \frac{1}{T}$.

In the case $T = 2\pi$, the Fourier series is $\sum_{n=-\infty}^{+\infty} \frac{\sin(n/2)}{n\pi}e^{inx} = \frac{1}{2\pi} + \sum_{n=1}^{n} \frac{2\sin(n/2)}{n\pi}\cos(nx)$.

Actually Sagemath has a built in method to calculate the Fourier series for us, as shown in the following code:

```
x = var('x') # Declare variable x
f = piecewise([[(-pi,-1/2),0],[(-1/2,1/2),1],[(1/2,pi),0] ] ) # Define f as a piecewise
function
FS5=f.fourier_series_partial_sum(5,pi) # For a piecewise function, there is a method called
"fourier_series_partial_sum", we calculate 5 terms
print(FS5) # print the result
P1 = f.plot() # plot the graph of f
P2 = plot(FS5,x,-pi,pi,linestyle="--") # plot the graph of fourier partial sum
(P1+P2).show(title="5 terms") # print the graph
FS100=f.fourier_series_partial_sum(50,pi)
P3 = plot(FS100,x,-pi,pi).show(title="100 terms") # What happens if we calculate 100 terms?
```

From the picture we can see that even though we approximate f hard enough, the approximation seems to be OK for points far from the jump. But near the jump we there is still big error remaining. This is called the Gibbs phenomenon.

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3. Pointwise convergence (simple case)

First observation as a consequence of Weierstrass M-test:

If $\sum_{n=-\infty}^{+\infty} |c_n| < \infty$, then $S(f)$ converges uniformly.

In this case we call the Fourier series of f *converge absolutely*. For a postive series, it converges when the coefficients decays sufficiently fast, and from calculus we know that $O(\frac{1}{n})$ is not enough and $O(\frac{1}{n^2})$ is enough. In fact, we can show using Fourier series of the quadratic function that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (in exercise).

Proposition 3.1 (The derivative theorem) Let f be continuously differentiable on $\mathbb{T}$ (this means f is continuously differentiable on $[0, 1]$ with $f(0) = f(1)$), then $\widehat{f'}(n) = 2\pi in\hat{f}(n)$.

Proof. This is just integration by part.
$\widehat{f'}(n) = \int_0^1 f'(x)e^{-2\pi inx}dx = \int_0^1 e^{-2\pi inx}df = f(x)e^{-2\pi inx}|_0^1 - \int_0^1 f(x)(-2\pi in)e^{-2\pi inx}dx = 2\pi in\int_0^1 f(x)e^{-2\pi inx}dx = 2\pi in\hat{f}(n)$
.

Remark. In the period T case $\widehat{f}(n) = \frac{2\pi}{T}in\hat{f}(n)$.

Lemma. (L^1 estimate) $|\hat{f}(n)| \leq \int_0^1 |f(x)|dx$. If we denote $\int_0^1 |f(x)|dx$ by $\|f\|_{L^1}$ then $|\widehat{f}(n)| \leq \|f\|_{L^1}$.

Proof. $|\hat{f}(n)| = |\int_0^1 f(x)e^{-2\pi inx}dx| \le \int_0^1 |f(x)|dx$.

Corollary. If f has continuous second derivative then $\widehat{f}(n) = O(\frac{1}{n^2})$, hence the Fourier series of f converges uniformly.

Proof. Apply the derivative theorem twice we get $\widehat{f''}(n) = -4\pi^2 n^2 \hat{f}(n)$. By the L^1 estimate we have $|\hat{f''}(n)| \le \int_0^1 |f''(x)|dx = \|f''\|_{L^1}$. So $|\hat{f}(n)| \le \frac{1}{4\pi}\|f''\| \cdot \frac{1}{n^2} = \text{constant} \cdot \frac{1}{n^2}$.

Next we consider the continuous case.

Proposition 3.2. Let f be an integrable function on the circle. If $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(x)$ vanishes at continuous points, i.e. for every $x \in [0, 2\pi]$ such that f is continuous at x, $f(x) = 0$.

Note. The Fourier coefficients vanishes means f is orthogonal to all trignomic polynomials.

Proof.

Method 1. Assume $f(x_0) \ne 0$ for some continuous point x_0. We can moreover assome $f(x_0) > 0$, then since f is continuous at x_0, it must be $\ge \epsilon$ near x_0. Then the idea is to construct a sequence of trignomial polynomials p_k such that p_k goes to $+\infty$ near x_0 and the negative part of p_k is controlled, so that $\int_0^{2\pi} f(t)p_k(t)dt \to \infty$, which contradicts to the assumption that f is orthogonal to p_k. Precisely, fix $\epsilon > 0$ and choose small $\delta > 0$ such that $f(x) \ge \epsilon$ on $(x_0 - \delta, x_0 + \delta)$. Let $p(x) = \cos(x - x_0) + \alpha$ for small $\alpha > 0$. Let $p_k(x) = p(x)^k$. We choose α so small that $|p(x)| < 1$ so $p_k(x) \to 0$ when $x \notin (x_0 - \delta, x_0 + \delta)$. Then $0 = \int_0^{2\pi} f(x)p_k(x)dx = \int_{x_0-\delta}^{x_0+\delta} f(x)p_k(x)dx + \int_{x \notin (x_0-\delta, x_0+\delta)} f(x)p_k(x)dx$. The first term go to $+\infty$ when $k \to \infty$, the second term go to 0 when $k \to \infty$, contradiction.

Note: the second limit is a consequence of the *dominated convergence theorem*. If you want a more elementary proof see the book Theorem 2.1.

The second method involves Lemma 5.1 and Theorem 5.2 in Chapter 2 of Stein-Shakarchi, that we'll do later.

Method 2. Let $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$ which is a trignomic polynomial called Fejer kernel. Assume f is continuous at 0, the Fejer theorem (later) implies $\frac{1}{2\pi}\int_0^{2\pi} f(x)F_N(x)dx \to f(0)$ when $N \to \infty$. Since f is orthogonal to trignomic polynomials, this shows $f(0) = 0$. If $x_0 \ne 0$, let $g(x) = f(x + x_0)$, apply the result to $g(x)$ shows $g(0) = 0 = f(x_0)$.

Corollary. Let f be an integrable function on the circle such that the Fourier series of f converges absolutely. If f is continuous at x_0, then the Fourier series of f converges to f at x_0, i.e. $\lim_{N\to\infty} S_N(f)(x_0) = f(x_0)$.

Proof. Since the Fourier series of f converges absolutely which implies $S(f)$ converges uniformly, hence $S(f)$ is a continuous function. Note that the Fourier coefficients of $S(f)$ is precisely $\hat{f}(n)$, so $S(f)$ and f has same Fourier coefficients. Then $S(f)$ and f coincide on continuous points by Proposition 3.2.

Appendix: Plot of Direchlet kernel and Fejer kernel

```
N=20
x=var('x')
DN=sin((N+1/2)*x)/sin(1/2*x)
FN=(1/N)*sin(N*x/2)^2/(sin(x/2)^2)
P1=plot(DN)
P1.show(title="Direchlet kernel")
P2=plot(FN)
P2.show(title="Fejer kernel")
```

We can also plot an approximate process for different values, for example the Fejer kernel

```
for N in [20,40,60,80]:
    FN=(1/N)*sin(N*x/2)^2/(sin(x/2)^2)
    P2=P2+plot(FN)
P2.show()
```