Exam

Problem 1

Let f be a continuous function on the circle and $\sum_{n=-\infty}^{+\infty} a_n e^{inx}$ its Fourier series.

1. (10) Show that $f = \sum_{n=-\infty}^{+\infty} a_n e^{inx}$ in the distributional sense.
   - *Hint: for a test function φ on the circle, write $\varphi(x) = \sum_{m=-\infty}^{+\infty} b_m e^{imx}$, show that* $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\varphi(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\sum_{n=-\infty}^{+\infty} a_n e^{inx})(\sum_{n=-\infty}^{+\infty} b_m e^{imx})dx$. Justify your solution carefully.
2. (10) Suppose f'' (the second derivative of f in the distributional sense) is equal to an L^2-function on the circle. Show that the Fourier series of f converges to f absolutely uniformly.

Problem 2

Given the function $f(t) = \cos(2\pi t)$.

1. (10) Compute the Fourier transform of f in the sense of tempered distributions.
2. (5) Let u be the tempered distribution $\sum_{k=-\infty}^{+\infty} \delta_{\frac{2k}{3}}$. Calculate $\mathcal{F}f * u$.
   - Hint: You can use the fact that $\delta_a * \delta_b = \delta_{a+b}$ and $\delta_a * u = \sum_{k \in \mathrm{Z}} \delta_{a+\frac{2}{3}k}$.
3. (5) Let $\Pi_{\frac{4}{3}}(x) = \begin{cases} 1, & -\frac{2}{3} < x < \frac{2}{3} \\ 0, & else \end{cases}$. Calculate $\Pi_{\frac{4}{3}} \cdot (\mathcal{F}f * u)$.
4. (5) Calculate $\mathcal{F}^{-1}[\Pi_{\frac{4}{3}} \cdot (\mathcal{F}f * u)]$.

Problem 3

(15) Let $f \in \mathcal{S}$ be bandlimited on $[-\frac{1}{2}, \frac{1}{2}]$, i.e. $\mathcal{F}f$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$. Show that $\int_{-\infty}^{+\infty} |f(x)|^2 dx$ = $\sum_{n \in \mathrm{Z}} |f(n)|^2$.

Problem 4

1. (10) Let $R(x_1, \ldots, x_n)$ be the n-dimensional rectangular function given by

$$R(x_1, \ldots, x_n) = \begin{cases} 1, & x \in Q; \\ 0, & \text{otherwise}, \end{cases}$$

where $Q := \{(x_1, \ldots, x_n) \in \mathrm{R}^n : -\frac{1}{2} < x_j < \frac{1}{2} \text{ for every } 1 \le j \le n\}$ is the n-dimensional unit square. Calculate the n-dimensional Fourier transform $\mathcal{F}R(\xi)$.

2. (10) Calculate $\int_{-\infty}^{+\infty} (\frac{\sin x}{x})^2 dx$.

Problem 5

Let $(G, +)$ be a finite abelian group and denote its zero-element by 0_G. Suppose f, g are complex-valued functions on G and the convolution of f and g is given by

$$f * g(a) = \frac{1}{|G|} \sum_{b \in G} f(a - b) g(b).$$

1. (5) Prove the convolution theorem $\widehat{f * g}(\chi) = \hat{f}(\chi)\hat{g}(\chi)$ for $\chi \in \hat{G}$.
2. Let $D : G \to \mathrm{C}$ be given by $D(c) = \sum_{\chi \in \hat{G}} \chi(c)$.
   - (5) Show that $D(c) = \begin{cases} |G|, & c = 0_G; \\ 0, & \text{otherwise.} \end{cases}$
   - (5) Show that $f * D = f$.
   - (5) The Fourier series of f is given by $Sf = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$. Show that $Sf = f * D$.

10'

1\. 1. A test function on the circle is a 2π-periodic smooth function φ with compact support on its periods. Then its Fourier series $\sum_{n=-\infty}^{+\infty} b_n e^{inx}$ converges uniformly.

Let $\sum_{n=-\infty}^{+\infty} a_n e^{inx}$ be the Fourier series of f, then

$$\int_{-\pi}^{\pi} f(x)\varphi(x)\,dx = \int_{-\pi}^{\pi} f(x) \sum_{m=-\infty}^{+\infty} b_m e^{imx}\,dx$$

$$= \int_{-\pi}^{\pi} \sum_{m=-\infty}^{+\infty} b_m f(x) e^{imx}\,dx$$

You need to explain that this is because uniform convergence.

$$= \sum_{m=-\infty}^{+\infty} b_m \int_{-\pi}^{\pi} f(x) e^{imx}\,dx$$

$$= \sum_{m=-\infty}^{+\infty} b_m a_{-m}$$

On the other hand, for positive integer N.

$$\int_{-\pi}^{\pi} \sum_{n=-N}^{N} a_n e^{inx} \sum_{m=-\infty}^{+\infty} b_m e^{imx}\,dx$$

$$= \sum_{n=-N}^{N} a_n \int_{-\pi}^{\pi} \sum_{m=-\infty}^{+\infty} b_m e^{i(n+m)x}\,dx$$

here we need uniform convergence

$$= \sum_{n=-N}^{N} a_n b_{-n}$$

Let $N\to\infty$ we get the desired result.

Note that the series converges because $|a_n|$ is uniformly bounded, $|b_n|\to 0$ repidly because φ is test fn

We have showed that $\int_{-\pi}^{\pi} f\varphi\,dx = \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{+\infty} a_n e^{inx}\right)\varphi(x)\,dx$.

10'

1.2. Since $f = \sum_{n=-\infty}^{+\infty} a_n e^{inx}$ in sense of distribution

$$f'' = \sum_{n=-\infty}^{+\infty} i^2 n^2 a_n e^{inx} \quad \text{in sense of distribution}$$

By assumption, f'' is L^2, so

$$\|f''\|^2_{L^2(S^1)} = \sum_{n=-\infty}^{+\infty} n^2 |a_n|^2 < \infty$$

This implies $n^2|a_n|^2$ is bounded.

$$\sum_{n=-\infty}^{+\infty} |a_n e^{inx}| \leq \sum_{n=-\infty}^{+\infty} |a_n| \leq \text{constant} \cdot \sum_{n=-\infty}^{+\infty} \frac{1}{n^2}$$

$$< \infty$$

So $\sum_{n=-\infty}^{+\infty} a_n e^{inx}$ converges to f uniformly by M test.

Problem 2.

10' 1. $f(t) = \cos(2\pi t) = \dfrac{e^{2\pi i t} + e^{-2\pi i t}}{2}$

$$\mathcal{F}f = \mathcal{F}\left(\frac{e^{2\pi i t} + e^{-2\pi i t}}{2}\right) = \frac{1}{2}(\delta_1 + \delta_{-1})$$

Note: $\mathcal{F}(e^{2\pi i t}) = \delta_1$ as tempered distribution

5' 2. $\mathcal{F}f * u = \frac{1}{2}(\delta_1 + \delta_{-1}) * \sum_{k=-\infty}^{+\infty} \delta_{\frac{2}{3}k}$

$$= \frac{1}{2}\sum_{k=-\infty}^{+\infty}\left(\delta_{1+\frac{2}{3}k} + \delta_{-1+\frac{2}{3}k}\right)$$

5' 3. Recall that $f \cdot \delta_a = f(a)\delta_a$ so

$$\Pi_{\frac{4}{3}} \cdot [\mathcal{F}f * u] = \frac{1}{2}\Pi_{\frac{4}{3}} \cdot \sum_{k=-\infty}^{+\infty}\left(\delta_{1+\frac{2}{3}k} + \delta_{-1+\frac{2}{3}k}\right)$$

$$= \frac{1}{2}\sum_{k=-\infty}^{+\infty}\left(\Pi_{\frac{4}{3}}\left(1+\tfrac{2}{3}k\right)\delta_{1+\frac{2}{3}k} + \Pi_{\frac{4}{3}}\left(-1+\tfrac{2}{3}k\right)\delta_{-1+\frac{2}{3}k}\right)$$

$\Pi_{\frac{4}{3}}\left(1+\frac{2}{3}k\right) = 1$ if and only if $-\frac{2}{3} < 1+\frac{2}{3}k < \frac{2}{3}$

$$\Leftrightarrow -\frac{5}{3} < \frac{2}{3}k < -\frac{1}{3}$$

$$\Leftrightarrow -5 < 2k < -1$$

$$\Leftrightarrow -2.5 < k < -0.5$$

i.e. $k = -2$ or $k = -1$

hence $\sum_{k=-\infty}^{+\infty} \Pi_{\frac{4}{3}}\left(1+\frac{2}{3}k\right)\delta_{1+\frac{2}{3}k} = \delta_{-\frac{1}{3}} + \delta_{\frac{1}{3}}$ $\leftarrow$ here we need the fact $f\delta_a = f(a)\delta_a$

Similarly, $\sum_{k=-\infty}^{+\infty} \Pi_{\frac{4}{3}}\left(-1+\frac{2}{3}k\right)\delta_{-1+\frac{2}{3}k} = \delta_{-\frac{1}{3}} + \delta_{\frac{1}{3}}$

It follows that $\Pi_{\frac{4}{3}} \cdot (\mathcal{F}f * u) = \delta_{-\frac{1}{3}} + \delta_{\frac{1}{3}}$

5'

4. $\mathcal{F}^{-1}\left[\Pi_{\frac{4}{3}} \cdot (\mathcal{F}f * u)\right] = \mathcal{F}^{-1}\left(\delta_{-\frac{1}{3}} + \delta_{\frac{1}{3}}\right)$

$$= e^{-2\pi i\left(-\frac{1}{3}x\right)} + e^{-2\pi i\left(\frac{1}{3}x\right)}$$

$$= e^{\frac{2}{3}\pi i x} + e^{-\frac{2}{3}\pi i x}$$

$$= 2\cos\left(\frac{2}{3}\pi x\right)$$

15'

Problem 3 Let $F(s)$ be the Fourier transform of f, i.e. $F(s) := \mathcal{F}f(s)$

Periodize F to a 1-periodic function F_1 i.e. $F_1(s) = \sum_{n=-\infty}^{+\infty} F(s-n)$

Then let $\sum_{n=-\infty}^{+\infty} \hat{F_1}(n) e^{inx}$ be the Fourier series of F_1

Note that since F is supported on $[-\frac{1}{2}, \frac{1}{2}]$

$$F = \Pi_1 \cdot F_1 \qquad (*)$$

Then we know that $\boxed{\hat{F_1}(n) = \mathcal{F}F(n) = (\mathcal{F}\mathcal{F}f)(n) = f(-n)}$

By parsval identity $\|F_1\|^2_{L^2(T)} = \sum_{n=-\infty}^{+\infty} |\hat{F_1}(n)|^2$

$= \sum_{n=-\infty}^{+\infty} |f(-n)|^2 = \sum_{n=-\infty}^{+\infty} |f(n)|^2$

By $(*)$ $\|F\|^2_{L^2(\mathbb{R})} = \|F_1\|^2_{L^2(T)}$

By Plancherel identity $\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\mathcal{F}f(s)|^2 ds$

$= \|F\|^2_{L^2(\mathbb{R})}$

Combine above identities we find

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |f(n)|^2$$

Problem 4

Note that $R(x_1,\cdots,x_n) = \Pi_1(x_1)\cdot\Pi_1(x_2)\cdots\Pi_1(x_n)$

10' 1. i.e. R is tensor product.

Then $\mathcal{F}R(\xi) = \mathcal{F}\Pi_1(\xi_1)\cdots\mathcal{F}\Pi_1(\xi_n)$

$$= \frac{\sin(\pi\xi_1)}{\pi\xi_1}\cdots\frac{\sin(\pi\xi_n)}{\pi\xi_n}$$

10' 2. Apply Plancherel to $\mathcal{F}\Pi_1(s) = \frac{\sin \pi s}{\pi s}$

we get $1 = \int_{-\infty}^{+\infty} |\Pi_1(t)|^2\,dt = \int_{-\infty}^{+\infty} \frac{|\sin(\pi s)|^2}{|\pi s|^2}\,ds$

let $x = \pi s$, we get $dx = \pi\,ds$

$$1 = \int_{-\infty}^{+\infty}\frac{\sin^2(\pi s)}{(\pi s)^2}\,ds = \int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}\cdot\frac{dx}{\pi}$$

$$\Rightarrow \int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}\,dx = \pi$$

Problem 5 $\quad \hat{f}(\chi) = \frac{1}{|G|}\int_G f\bar{\chi} = \frac{1}{|G|}\sum_{a\in G} f(a)\chi(-a)$

5' $\quad$ 1. $\widehat{f*g}(\chi) = \frac{1}{|G|}\sum_{a\in G} f*g(a)\,\chi(-a)$

$$= \frac{1}{|G|}\sum_{a\in G}\left(\frac{1}{|G|}\sum_{b\in G} f(a-b)g(b)\right)\chi(-a)$$

$$= \frac{1}{|G|^2}\sum_{b\in G}\sum_{a\in G} f(a-b)g(b)\chi(-a+b)\chi(-b)$$

$$= \frac{1}{|G|}\sum_{b\in G} g(b)\chi(-b)\cdot\frac{1}{|G|}\sum_{a\in G} f(a-b)\chi(-(a-b))$$

$$= \hat{g}(\chi)\,\hat{f}(\chi)$$

5' $\quad$ (2) $D(0) = \sum_{\chi\in\hat{G}}\chi(0) = \sum_{\chi\in\hat{G}} 1 = |\hat{G}| = |G|$

here we need Thm 2.5 of chapter 7

for $c\neq 0_G$. $D(c) = \sum_{\chi\in\hat{G}}\chi(c)$

fix any χ_0 s.t $\chi_0(c)\neq 0$. $D(c)\cdot\chi_0(c) = \sum_{\chi\in\hat{G}}\chi(c)\cdot\chi_0(c) = \sum_{\chi\in\hat{G}}(\chi+\chi_0)(c)$

The only possibility for this to hold is $D(c)=0$. $\quad = \sum_{\chi'\in\hat{G}}\chi'(c) = D(c)$

5' $\quad (f*D)(a) = \frac{1}{|G|}\sum_{b\in G} f(a-b)D(b)$

show $f*D=f$

$$= \frac{1}{|G|} f(a-0)\cdot|G| = f(a)$$

5' show $Sf = f*D$

$Sf(a) = \sum_{\chi\in\hat{G}}\hat{f}(\chi)\chi(a) = \frac{1}{|G|}\sum_{\chi\in\hat{G}}\sum_{b\in G} f(b)\chi(-b)\chi(a)$

$$= \frac{1}{|G|} \sum_{b \in G} \sum_{\chi \in \hat{G}} f(b)\, \chi(a-b)$$

$$= \frac{1}{|G|} \sum_{b \in G} f(b) \cdot D(a-b)$$

$$= \frac{1}{|G|} f(a) \cdot |G| = f(a)$$

We have showed $Sf(a) = (f * D)(a) = f(a)$.