Exam

Problem 1

Let f be a continuous function on the circle and $\sum_{n=-\infty}^{+\infty} a_n e^{inx}$ its Fourier series.

1. (10) Let $S_N := \sum_{n=-N}^{N} a_n e^{inx}$. Show that $f = \lim_{N\to\infty} S_N$ in the distributional sense.
   - *Hint: for a test function φ on the circle, write $\varphi(x) = \sum_{m=-\infty}^{+\infty} b_m e^{imx}$, show that* $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\varphi(x)dx = \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} S_N(x)(\sum_{m=-\infty}^{+\infty} b_m e^{imx})dx$. Justify your solution carefully.
2. (10) Suppose f'' (the second derivative of f in the distributional sense) is equal to an L^2-function on the circle. Show that the Fourier series of f converges to f absolutely uniformly.

Problem 2

Given the function $f(t) = \cos(2\pi t)$.

1. (10) Compute the Fourier transform of f in the sense of tempered distributions.
2. (5) Let u be the tempered distribution $\sum_{k=-\infty}^{+\infty} \delta_{\frac{2k}{3}}$. Calculate $\mathcal{F}f * u$.
   - Hint: You can use the fact that $\delta_a * \delta_b = \delta_{a+b}$ and $\delta_a * u = \sum_{k\in\mathbb{Z}} \delta_{a+\frac{2}{3}k}$.
3. (5) Let $\Pi_{\frac{4}{3}}(x) = \begin{cases} 1, -\frac{2}{3} < x < \frac{2}{3} \\ 0, else \end{cases}$. Calculate $\Pi_{\frac{4}{3}} \cdot (\mathcal{F}f * u)$.
4. (5) Calculate $\mathcal{F}^{-1}[\Pi_{\frac{4}{3}} \cdot (\mathcal{F}f * u)]$.

Problem 3

(15) Let $f \in \mathcal{S}$ be bandlimited on $[-\frac{1}{2}, \frac{1}{2}]$, i.e. $\mathcal{F}f$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$. Show that $\int_{-\infty}^{+\infty} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |f(n)|^2$.

Problem 4

1. (10) Let $R(x_1, \ldots, x_n)$ be the n-dimensional rectangular function given by $R(x_1, \ldots, x_n) = \begin{cases} 1, & x \in Q; \\ 0, & \text{otherwise,} \end{cases}$ where $Q := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : -\frac{1}{2} < x_j < \frac{1}{2} \text{ for every } 1 \le j \le n\}$ is the n-dimensional unit square. Calculate the n-dimensional Fourier transform $\mathcal{F}R(\xi)$.
2. (10) Calculate $\int_{-\infty}^{+\infty} (\frac{\sin x}{x})^2 dx$.

Problem 5

Let $(G, +)$ be a finite abelian group and denote its zero-element by 0_G. Suppose f, g are complex-valued functions on G and the convolution of f and g is given by

$f * g(a) = \frac{1}{|G|} \sum_{b\in G} f(a-b)g(b)$.

1. (5) Prove the convolution theorem $\widehat{f * g}(\chi) = \hat{f}(\chi)\hat{g}(\chi)$ for $\chi \in \hat{G}$.

2. Let $D : G \to \mathbb{C}$ be given by $D(c) = \sum_{\chi \in \hat{G}} \chi(c)$.

- (5) Show that $D(c) = \begin{cases} |G|, & c = 0_G; \\ 0, & \text{otherwise.} \end{cases}$
- (5) Show that $f * D = f$.
- (5) The Fourier series of f is given by $Sf = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi$. Show that $Sf = f * D$.