SUMMARY AND SOLUTIONS TO SOME EXERCISES

1. FOURIER SERIES

The key fact is the following (See Theorem 1.2, 1.3):

$\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-\pi,\pi]$.

The above fact implies that: for a function f in $L^2[-\pi,\pi]$ (sometimes we also write $L^2[-\pi,\pi]$ as $L^2(S^1)$) its *formal* Fourier series expansion

$$f \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}, \quad \hat{f}(n) := (f, e^{inx}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx,$$

converges in L^2 sense:

$$\lim_{N\to\infty} ||f - f_N|| = 0, \quad f_N := \sum_{|n|\le N} \hat{f}(n)e^{inx}.$$

The proof of the above fact depends on our main theorem (Theorem 1.1), which says that all piecewise smooth 2π-periodic functions satisfy the following "pointwise" convergence property:

$$\lim_{N\to\infty} \left|f_N(x) - \frac{f(x+)+f(x-)}{2}\right| = 0, \quad \forall\, x\in\mathbb{R}.$$

How to apply it ? Three steps!

Step 1: For a function f on $[a,b]$, which we assume that $b-a=2\pi$ (otherwise use a change of variable), verify that it is piecewise smooth then use "integration by parts" to compute

$$\hat{f}(n) = \frac{1}{2\pi}\int_a^b f(x)e^{-inx}\,dx.$$

Step 2: Extend f to a piecewise smooth 2π-periodic function, still denote it by f, compute right limits $f(x+)$ and left limits $f(x-)$.

Step 3: Apply our main theorem.

Examples: Exercise 4 (Week 3) and page 11 of the lecture notes. In page 11, we look at (for every fixed $s\in\mathbb{C}\setminus\mathbb{Z}$)

$$f(x) = \frac{\pi}{\sin\pi s}e^{i(\pi-x)s}, \quad x\in(0,2\pi),$$

and want to prove

$$\lim_{N\to\infty}\sum_{|n|\le N}\frac{1}{n+s} = \pi\cot\pi s, \quad \forall\, s\in\mathbb{C}\setminus\mathbb{Z}.$$

Step 1: If $s\in\mathbb{C}\setminus\mathbb{Z}$ then $\sin\pi s\neq 0$, thus f is smooth on a neighborhood of $[0,2\pi]$ and

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi}\frac{\pi}{\sin\pi s}e^{i(\pi-x)s}e^{-inx}\,dx = \frac{e^{i\pi s}}{2\sin\pi s}\int_0^{2\pi} e^{-i(n+s)x}\,dx,$$

notice that $s \in \mathbb{C} \setminus \mathbb{Z}$ implies that $n+s$ can never be zero, hence

$$e^{-i(n+s)x} = \left(\frac{e^{-i(n+s)x}}{-i(n+s)} \right)',$$

thus Newton-Lebniz formula gives

$$\int_0^{2\pi} e^{-i(n+s)x}\, dx = \left(\frac{e^{-i(n+s)x}}{-i(n+s)} \right) \Big|_0^{2\pi} = \frac{e^{-2\pi i s} - 1}{-i(n+s)}.$$

Now we have

$$\hat{f}(n) = \frac{e^{i\pi s}}{2 \sin \pi s} \cdot \frac{e^{-2\pi i s} - 1}{-i(n+s)} = \frac{1}{n+s} \cdot \frac{1}{\sin \pi s} \cdot \frac{e^{i\pi s} - e^{-i\pi s}}{2i}.$$

Since the Euler formula gives $e^{i\pi s} - e^{-i\pi s} = 2i \sin \pi s$, the above formula reduces to

$$\hat{f}(n) = \frac{1}{n+s}.$$

Step 2 : Notice that the right limit $f(0+) = \frac{\pi}{\sin \pi s} e^{i\pi s}$ and the left limit $f(2\pi-) = \frac{\pi}{\sin \pi s} e^{-i\pi s}$, if we extend f from $(0, 2\pi)$ to a 2π periodic function on $\mathbb{R}$ then we must have $f(0-) = f(2\pi-) = \frac{\pi}{\sin \pi s} e^{-i\pi s}$, which gives

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{\sin \pi s} \frac{e^{i\pi s} + e^{-i\pi s}}{2} = \pi \frac{\cos \pi s}{\sin \pi s} = \pi \cot \pi s.$$

Step 3 : Now apply our main theorem at $x = 0$ to the extended function, we get

$$\lim_{N \to \infty} \sum_{|n| \le N} \frac{1}{n+s} = \pi \cot \pi s, \quad \forall\, s \in \mathbb{C} \setminus \mathbb{Z}. \tag{1}$$

This identity also has many applications. For example, in Exercise 4, Week 10-11,

$$\mathcal{D}_R(x) = \frac{\sin 2\pi R x}{\pi x},$$

thus when R is a positive integer and $x \notin \mathbb{Z}$ we have

$$\sum_{|n| \le N} \mathcal{D}_R(x+n) = \sum_{|n| \le N} \frac{\sin 2\pi R(x+n)}{\pi(x+n)} = \sum_{|n| \le N} \frac{\sin 2\pi R x}{\pi(x+n)} = \frac{\sin 2\pi R x}{\pi} \sum_{|n| \le N} \frac{1}{n+x}.$$

Hence (1) gives

$$\lim_{N \to \infty} \sum_{|n| \le N} \mathcal{D}_R(x+n) = \frac{\sin 2\pi R x}{\pi} \pi \cot \pi x = \frac{\sin 2\pi R x \cos \pi x}{\sin \pi x},$$

since $\sin 2\pi R x \cos \pi x = \sin(2R-1)\pi x + \cos 2\pi R x \sin \pi x$, we get

$$\lim_{N \to \infty} \sum_{|n| \le N} \mathcal{D}_R(x+n) = \frac{\sin(2R-1)\pi x}{\sin \pi x} + \cos 2\pi R x.$$

By the Dirichlet kernel formula (page 4 of the notes), we have

$$\frac{\sin(2R-1)\pi x}{\sin \pi x} = \sum_{|n|\leq R-1} e^{2\pi i x},$$

together with $\cos 2\pi Rx = (e^{2\pi iRx}+e^{-2\pi iRx})/2$ we solved Exercise 4, Week 10-11 in case $x \notin \mathbb{Z}$. But when $x \in \mathbb{Z}$, the exercise is trivial.

2. Fourier transform

Basic facts (you MUST know the proof):

1. Definition of the Schwartz space $\mathcal{S}$:

$$f \in \mathcal{S} \Leftrightarrow f \in C^\infty(\mathbb{R}),\ \sup_{x\in\mathbb{R}} |x^k f^{(l)}(x)| < \infty, \forall\, k, l \geq 0.$$

2. Fourier inversion formula:

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi,\quad \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}\, dx,\ \ \forall\, f \in \mathcal{S}.$$

3. Plancherel identity:

$$(\hat{f}, \hat{g}) = (f, g),\ \ \forall\, f, g \in \mathcal{S}.$$

4. Canonical fixed point of the Fourier transform:

$$\widehat{e^{-\pi x^2}} = e^{-\pi\xi^2}.$$

5. Definition of the tempered distribution: a linear functional $T : \mathcal{S} \to \mathbb{C}$ defines a tempered distribution (we say that $T \in \mathcal{S}'$) if there exist $C > 0, N \in \mathbb{N}$ such that

$$|T(f)| \leq C \sup_{x\in\mathbb{R},k,l\leq N} |x^k f^{(l)}(x)|,\ \ \forall\, f \in \mathcal{S}.$$

Fourier transform of T is also a tempered distribution defined by

$$\hat{T}(f) := T(\hat{f}),\ \ \forall\, f \in \mathcal{S}.$$

Remark: One may use tempered distribution to rewrite the Fourier inversion formula as

$$\hat{1} = \delta_0.$$

6. Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),\ \ \forall\, f \in \mathcal{S}.$$

In the lecture notes, we use the Poisson summation formula to prove the Fourier inversion formula and the Plancherel identity, and we use our main theorem in Fourier series to prove the Poisson summation formula.

3. Wavelets

We start from the Haar scaling function ϕ

$$\phi := 1 \text{ on } [0,1); \quad \phi := 0 \text{ otherwise};$$

the associated Haar wavelet function ψ is defined by

$$\psi(x) = \phi(2x) - \phi(2x-1).$$

The fundamental fact (you should know the proof) is

$$\{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$$

defines an orthonormal system (in fact it is also an orthonormal basis) of $L^2(\mathbb{R})$. In order to find a continuous scaling function, we introduced the "multiresolution analysis" theory. An important example is the following:

Shannon multiresolution analysis: Exercise 5.8 from Boggess–Narcowich. The Shannon scaling function is defined by

$$\phi(x) := \frac{\sin \pi x}{\pi x}.$$

A basis fact is: ϕ is the Fourier transform of the indicator function $1_{[-1/2,1/2]}$ since

$$\int_{\mathbb{R}} 1_{[-1/2,1/2]}(y) e^{-2\pi i x y}\, dy = \int_{-1/2}^{1/2} e^{-2\pi i x y}\, dy = \frac{e^{-\pi i x} - e^{\pi i x}}{-2\pi i x} = \frac{\sin \pi x}{\pi x}.$$

Now since $\{1_{[-1/2,1/2]} e^{2\pi i k x}\}_{k\in\mathbb{Z}}$ defines an orthornormal basis of $L^2[-1/2,1/2] \subset L^2(\mathbb{R})$, we know that

$$\{\widehat{1_{[-1/2,1/2]} e^{2\pi i k x}}\}_{k\in\mathbb{Z}}$$

must defines an orthornormal basis of $L^2\widehat{[-1/2,1/2]}$ by the Plancherel identity. By the Fourier inversion formula, we can write $L^2\widehat{[-1/2,1/2]}$ as

$$V_0 := \{f \in L^2(\mathbb{R}) : \hat{f} \equiv 0 \text{ outside } [-1/2,1/2]\}.$$

On the other hand, a direct computation gives

$$\widehat{1_{[-1/2,1/2]} e^{2\pi i k y}} = \frac{\sin \pi(x-k)}{\pi(x-k)} = \phi(x-k).$$

Thus we get (equivalent to the Sampling theorem, see Theorem 2.16)

$$V_0 = L^2\widehat{[-1/2,1/2]} = \overline{\text{Span}\{\phi(x-k)\}_{k\in\mathbb{Z}}}.$$

To get the related multiresolution theory, one can continue to define V_j (by a simply dilation on V_0) and find the associated Shannon wavelet function.

The other fundamental fact is the Wavelet transform formula, the key is the *wavelet Plancherel identity* (Theorem 9 in the second part of the notes, you should try to understand the proof).

4. SPECIFIC TOPIC: EIGENTHEORY OF THE FOURIER TRANSFORM

The starting point again is $\widehat{e^{-\pi x^2}} = e^{-\pi \xi^2}$.

You may first go through page 1-4 of the second part of the lecture notes. The key fact is

Hermitie functions define an orthogonal basis of the Schwartz space $\mathcal{S}$ (and $L^2(\mathbb{R})$).

The idea is to consider another self-adjoint operator $\mathfrak{f}$ which satisfies

$$\mathfrak{f}(h_k) = -\left(k + \frac{1}{2}\right) h_k,$$

Since $\mathfrak{f}$ is self-adjoint we have

$$-\left(j + \frac{1}{2}\right)(h_j, h_k) = (\mathfrak{f} h_j, h_k) = (h_j, \mathfrak{f} h_k) = -\left(k + \frac{1}{2}\right)(h_j, h_k),$$

which gives $(h_j, h_k) = 0$ for $j \neq k$. The completeness of $\{h_k\}$ is proved in Lemma 3, in which the key is to consider the generating function

$$H(t, x) := \sum_{k=0}^{\infty} h_k(x) \frac{t^k}{k!},$$

which satisfies (see Lemma 2)

$$H(t, x) = e^{-\pi(x+2t)^2} e^{2\pi t^2}.$$

Related exercises are Week 8, 9. The followings are examples.

Exercise 1 (week 9): One hand since $\{h_k\}$ is orthogonal we have

$$\int_{\mathbb{R}} H(t, x)^2 \, dx = \sum_{k=0}^{\infty} ||h_k||^2 \frac{t^{2k}}{(k!)^2},$$

on the other hand, a direct computation gives

$$\int_{\mathbb{R}} H(t, x)^2 \, dx = \int_{\mathbb{R}} e^{-2\pi(x+2t)^2} e^{4\pi t^2} \, dx = e^{4\pi t^2} 2^{-1/2}.$$

Now it suffices to compare the Taylor coefficients.

Exercise 2 (week 9): It is enough to prove that the adjoint of $A - B$ is $-(A + B)$, i.e.

$$(f, (A - B)g) = (-(A + B)f, g), \quad \forall\, f, g \in \mathcal{S}.$$

(e g consider $f = (A - B)h_{k-1}, B = h_{k-1}$). Notice that the adjoint of B is B, thus it is enough to prove the adjoint of A is $-A$, i.e.

$$\int_{\mathbb{R}} f' \bar{g} \, dx = -\int_{\mathbb{R}} f \bar{g}' \, dx,$$

which follows from $\int_{\mathbb{R}} f'\bar{g} + f\bar{g}' \, dx = \int_{\mathbb{R}} (f\bar{g})' \, dx = \lim_{N \to \infty} ((f\bar{g})(N) - (f\bar{g})(-N)) = 0$ since f, g are in $\mathcal{S}$ (decreasing very fast).