J. Fourier, 1808-9: "Regarding the researches of d'Alembert and Euler could one not add that if they know this expansion, they made but a very imperfect use of it. They were both persuaded that an arbitrary and discontinuous function could never be resolved in series of this kind, and it does not even seem that anyone had developed a constant in cosines of mulitiple arcs, the first probelm which I had to solve in the theory of heat."

FOURIER ANALYSIS

XU WANG

These notes (written from 2018-10-24) for TMA4170 are based on Dym–Mckean's monograph [3], Stein–Shakarchi's book [9] and Boggess–Narcowich's book [1].

The purpose of the notes is to give a mathematical account of Fourier ideas on the circle and the line. The emphasis is placed on the applications, which include

1). Eigenfunction expansion for the Dirac operator $-i\frac{d}{dx}$;
2). Wirtinger and Poincaré inequality;
3). Heat equation on the circle;
4). Weyl's equidistribution theorem;
5). Random walks;
6). Poisson summation formula;
7). Jacobi theta identities;
8). Paley–Wiener theorem;
9). Sampling theorem;
10). Heisenberg uncertainty principle;
11). Polynomial approximation;
12). Gibbs' phenomenon;
13). Central limit theorem;
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CONTENTS

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1. Fourier series

1.1. **Main definition and some examples.** We shall follow Stein-Shakarchi's book in this section. Fix $a \in \mathbb{R}, L > 0$. Let f be a continuous function on $[a, a+L]$.

Definition 1.1. *The n-th Fourier coefficient of f is defined by*

$$\hat{f}(n) = \frac{1}{L}\int_a^{a+L} f(x) e^{-2\pi i n x/L}\, dx, \qquad n \in \mathbb{Z}.$$

The Fourier series of f is formally given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x/L}.$$

Remark: At this point, we do not say anything about the convergence of the series. The followings are examples from page 36–38 in [9]:

Example 1: Let $f(x) = x$ for $x \in [-\pi, \pi]$. We have

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\overline{e^{inx}}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\, dx.$$

Since

$$(e^{-inx})' = (-in)e^{-inx},$$

if $n \neq 0$ we have (by integration by parts)

$$\int_{-\pi}^{\pi} x e^{-inx}\, dx = \int_{-\pi}^{\pi} x \left(\frac{e^{-inx}}{-in}\right)' dx = x\left(\frac{e^{-inx}}{-in}\right)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{e^{-inx}}{-in}\right) dx = \frac{2\pi(-1)^n}{-in},$$

which gives

$$\hat{f}(n) = \frac{(-1)^{n+1}}{in}, \qquad n \neq 0.$$

If $n = 0$ then

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

Thus

$$f(x) \sim \sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{inx}.$$

Example 2: Fix $s \in \mathbb{C}\backslash\mathbb{Z}$ and consider

$$f(x) = \frac{\pi}{\sin \pi s} e^{i(\pi - x)s}, \qquad 0 \leqslant x \leqslant 2\pi.$$

By definition, we have

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi}{\sin \pi s} e^{i(\pi-x)s}\overline{e^{inx}}\,dx = \frac{e^{i\pi s}}{2\sin \pi s}\int_0^{2\pi} e^{-i(n+s)x}\,dx,$$

integration by parts gives

$$\int_0^{2\pi} e^{-i(n+s)x}\,dx = \frac{e^{-i(n+s)x}}{-i(n+s)}\Big|_0^{2\pi} = \frac{e^{-2i\pi s} - 1}{-i(n+s)}.$$

Apply the Euler formula $e^{ix} = \cos x + i \sin x$ (see the appendix), we get

$$\hat{f}(n) = \frac{1}{n+s}$$

and

$$f(x) \sim \sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}.$$

Later we shall see that the right hand side is closely related to the *Green function* of $-i\frac{d}{dx} + s$ on $\mathbb{R}/2\pi\mathbb{Z}$, where $-i\frac{d}{dx}$ is also called the *one dimensional Dirac operator*.

Example 3: The trigonometric polynomial defined for $x \in [-\pi, \pi]$ by

$$D_N(x) := \sum_{n=-N}^{N} e^{inx}$$

is called the N-th *Dirichlet kernel* and is of fundamental importance in the theory (as we shall see later). A closed formula for the Dirichlet kernel is

$$D_N(x) = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}, \ \ 0 < |x| < \pi; \quad D_N(x) = 2N+1, \ \ x = 0. \tag{1}$$

This can be seen by summing the geometric progressions $\sum_{n=0}^{N} \omega^n$ and $\sum_{n=-N}^{-1} \omega^n$ with $\omega = e^{ix}$. These sums are, respectively, equal to

$$\frac{1-\omega^{N+1}}{1-\omega}, \ \text{ and } \ \frac{1-\omega^{-N-1}}{1-\omega^{-1}} - 1.$$

Their sum is then

$$\frac{\omega^{-N} - \omega^{N+1}}{1-\omega} = \frac{\omega^{-N-\frac{1}{2}} - \omega^{N+\frac{1}{2}}}{\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}}} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}},$$

given the desired result.

Example 4: The function $P_r(\theta)$, called the *Poisson kernel*, is defined for $\theta \in [-\pi, \pi]$ and $0 \leqslant r < 1$ by the absolutely and uniformly convergent series

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}.$$

Exercise: Check that the n-th Fourier coefficient of P_r is $r^{|n|}$ and

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

1.2. **The first question.** Let us start form the following fact: *all trigonometric polynomials are equal to their Fourier series*. In fact, if f is a degree N trigonometric polynomial, i e

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

on $[a, a+2\pi]$, then the lemma below implies

$$\hat{f}(n) = c_n, \ \ |n| \leqslant N; \quad \hat{f}(n) = 0, \ \ |n| > N.$$

Lemma 1.2. *If $m \neq n$ then*

$$\int_a^{a+2\pi} e^{imx}\overline{e^{inx}}\, dx = \int_a^{a+2\pi} e^{i(m-n)x}\, dx = 0;$$

if $m = n$ then

$$\int_a^{a+2\pi} e^{imx}\overline{e^{inx}}\, dx = \int_a^{a+2\pi} 1\, dx = 2\pi.$$

We hope that a large class of functions have Fourier series expansion. A few reflections lead us to study the following question:

Let f be a 2π-periodic function on $\mathbb{R}$. Find a natural condition on f such that all

$$\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{e^{inx}}\, dx$$

are well defined and the following series $\{f_N(x)\}_{N\geqslant 0}$ defined by

$$f_N(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

converges (in a certain sense) to $f(x)$.

1.3. **Pointwise convergence.** Denote by $C^k(S^1)$ the space of all 2π-periodic C^k functions on $\mathbb{R}$, we shall use the following extension of $C^k(S^1)$.

Definition 1.3. *A 2π-periodic function f on $\mathbb{R}$ is said to be piecewise C^k $(k = 0, 1, \cdots)$ if there exists*

$$-\pi = x_0 < x_1 < \cdots < x_{m-1} < x_m = \pi, \qquad m \geqslant 1,$$

such that for each $0 < j \leqslant m-1$, $f|_{(x_j, x_{j+1})}$ extends to a C^k function on a neighborhood of $[x_j, x_{j+1}]$. We shall denote by $PC^k(S^1)$ the space of all 2π-periodic piecewise C^k functions.

Remark: Let $f \in PC^0(S^1)$. Then all its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} f(x) \overline{e^{inx}} \, dx$$

are well defined. Lemma 1.2 suggests to define:

Definition 1.4. *We call*

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx,$$

the inner product on $PC^0(S^1)$ *and write* $||f|| = (f, f)^{\frac{1}{2}}$.

Remark: It is clear that

$$\hat{f}(n) = (f, e^{inx}), \qquad \forall \, f \in PC^0(S^1),$$

which gives

$$f_N(x_0) = (f, D_N(x - x_0)). \tag{2}$$

We shall prove

Theorem 1.5. *If* $f \in PC^1(S^1)$ *then*

$$\lim_{N \to \infty} \left| f_N(x) - \frac{f(x+) + f(x-)}{2} \right| = 0. \tag{3}$$

Remark: For a general function $f \in PC^0(S^1)$, we have

$$\lim_{N \to \infty} \left| \frac{f_0(x) + \cdots + f_{N-1}(x)}{N} - \frac{f(x+) + f(x-)}{2} \right| = 0,$$

the proof is given in Exercise set 1 (see the end of the notes).

1.4. **Proof of Theorem 1.5 by Chernoff** [2].

1.4.1. *Chernoff identity.* Fix $x_0 \in \mathbb{R}$, assume further that f is C^1 near x_0. Then

$$g(x - x_0) = \frac{f(x) - f(x_0)}{e^{i(x - x_0)} - 1}$$

lies in $PC^0(S^1)$ and $g(y)$ is continuous near $y = 0$ (try!). Notice that

$$f(x) = (e^{i(x - x_0)} - 1) g(x - x_0) + f(x_0).$$

Change of variable $y = x - x_0$ gives the following *Chernoff identity* (try!)

$$f_N(x_0) - f(x_0) = \hat{g}(-N - 1) - \hat{g}(N).$$

1.4.2. *Bessel inequality.* Apply Lemma 1.2, we get

$$(g_N, g) = (g_N, g_N).$$

Thus we know that g_N is orthogonal to $g - g_N$, i.e.

$$(g_N, g - g_N) = 0,$$

which gives the following *Bessel inequality*

$$||g||^2 = ||g - g_N||^2 + ||g_N||^2 \geqslant ||g_N||^2 = \sum_{|n| \leqslant N} |\hat{g}(n)|^2.$$

Remark: The Bessel inequality

$$||g||^2 \geqslant \sum_{|n| \leqslant N} |\hat{g}(n)|^2$$

is true for every $g \in PC^0(S^1)$.

1.4.3. *Riemann–Lebesgue lemma.* The Bessel inequality gives the following *Riemann–Lebesgue lemma*

$$|\hat{g}(n)| \to 0, \qquad \text{as } |n| \to \infty,$$

which proves (3) in case f is C^1 at x.

Remark: The Riemann–Lebesgue lemma

$$|\hat{g}(n)| \to 0, \qquad \text{as } |n| \to \infty,$$

is true for every $g \in PC^0(S^1)$.

1.4.4. *Using Dirichlet kernel for the general case.* Recall that

$$f_N(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0 + x) D_N(x)\, dx.$$

Since D_N is even and $f(x_0 + x)$ can be written as

$$f(x_0 + x) = f_1(x) + f_2(x),$$

where

$$f_1(x) := \frac{f(x_0 + x) + f(x_0 - x)}{2}$$

is even and

$$f_2(x) := \frac{f(x_0 + x) - f(x_0 - x)}{2}$$

is odd, we have

$$f_N(x_0) = (f_1, D_N).$$

Notice that (try!) $f \in PC^1(S^1)$ implies that f_1 is C^1 near x_0, thus the previous argument gives

$$\lim_{N \to \infty} |f_N(x_0) - f_1(0)| = 0.$$

Since $f_1(0) = \frac{f(x_0+)+f(x_0-)}{2}$, the proof of Theorem 1.5 is complete.

1.5. **Mean square convergence.**

Theorem 1.6. *For every $f \in C^1(S^1)$, we have*

$$|f_N(x) - f(x)|^2 \leqslant ||f'||^2 \cdot \frac{2}{N}, \qquad \forall\, x \in \mathbb{R},$$

in particular $||f_N - f||^2 \leqslant ||f'||^2 \cdot \frac{2}{N}$

Proof. Let $N' > N$, we have

$$|f_N(x) - f_{N'}(x)| = |\sum_{|n|=N+1}^{N'} \hat{f}(n)e^{inx}| \leqslant \sum_{|n|=N+1}^{N'} |\hat{f}(n)|.$$

Notice that if $n \neq 0$ then

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{e^{-inx}}{-in}\right)'\,dx.$$

Thus integration by parts gives

$$\hat{f}(n) = \frac{-i}{n} \cdot \widehat{f'}(n).$$

Now we have

$$|f_N(x) - f_{N'}(x)| \leqslant \sum_{|n|=N+1}^{N'} |\widehat{f'}(n)| \cdot \frac{1}{n}.$$

Thus the Cauchy-Schwarz inequality gives

$$|f_N - f_{N'}|^2 \leqslant \left(\sum_{|n|=N+1}^{N'} |\widehat{f'}(n)|^2\right) \cdot \left(\sum_{|n|=N+1}^{N'} \frac{1}{n^2}\right).$$

Since

$$\sum_{|n|=N+1}^{N'} \frac{1}{n^2} = 2\sum_{n=N+1}^{N'} \frac{1}{n^2} \leqslant 2\int_N^{\infty} \frac{dx}{x^2} = \frac{2}{N}$$

and Bessel's inequality gives

$$\sum_{|n|=N+1}^{N'} |\widehat{f'}(n)|^2 \leqslant ||f'||^2,$$

we have

$$|f_N(x) - f_{N'}(x)|^2 \leqslant ||f'||^2 \cdot \frac{2}{N},$$

thus the theorem follows from Theorem 1.5 by letting $N' \to \infty$. □

Remark: Since $\{e^{inx}\}_{n\in\mathbb{Z}}$ satisfies

$$(e^{inx}, e^{inx}) = 1, \;\; (e^{inx}, e^{imx}) = 0, \;\; n \neq m,$$

we know that $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an *orthonormal family* in $C^1(S^1)$. The above theorem says that every element in $C^1(S^1)$ can be *approximated* by finite sums $\sum_{|n|\leqslant N} c_n e^{inx}$ generated by $\{e^{inx}\}_{n\in\mathbb{Z}}$, thus we know that

Theorem 1.7. *$\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $C^1(S^1)$.*

Another consequence of Theorem 1.6 is the following

Theorem 1.8 (Parseval's identity)**.** $||f||^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$ *for every $f \in C^1(S^1)$.*

1.5.1. *Completion of $C^1(S^1)$.* Let us recall the following definitions

Definition 1.9. *$\{g_n\}_{n\in\mathbb{N}} \in C^1(S^1)$ is said to be a Cauchy sequence if for every j there exists N_j such that*

$$||g_n - g_m|| < \frac{1}{j}, \qquad \forall\, n, m \geqslant N_j.$$

Two Cauchy sequences $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are said to be equivalent if for every j there exists N_j such that

$$||g_n - h_n|| < \frac{1}{j}, \qquad \forall\, n \geqslant N_j.$$

Denote by $[\{g_n\}]$ the set of Cauchy sequences equivalent to $\{g_n\}$, we call $[\{g_n\}]$ the equivalent class of $\{g_n\}$.

One may check that (try!)

$$([\{g_n\}], [\{h_n\}]) := \lim_{n\to\infty}(g_n, h_n)$$

is well defined.

Definition 1.10. *We call the set of all equivalent classes of Cauchy sequences in $C^1(S^1)$ with the above inner product the completion of $C^1(S^1)$ and denote it by $L^2(S^1)$ or $L^2[-\pi, \pi]$.*

Remark: Notice that

$$f \mapsto [\{f, f, \cdots, \}]$$

defines an injective map from $C^1(S^1)$ to $L^2(S^1)$. Thus we may look at $C^1(S^1)$ as a subset in $L^2(S^1)$. We leave it as an *exercise* to check that $C^1(S^1)$ is dense in $L^2(S^1)$. Thus Theorem 1.6 implies

Theorem 1.11. *$L^2(S^1)$ is a* separable complex Hilbert space *with orthonormal basis $\{e^{inx}\}_{n\in\mathbb{Z}}$.*

Remark: Anther way to look at $L^2[-\pi, \pi]$ is to use the *Lebesgue integral* theory (see Appendix 2), which gives the following isomorphism

$$L^2[-\pi, \pi] \simeq \{f \in \mathcal{M}[-\pi, \pi] : \int_{-\pi}^{\pi} |f|^2 dx < \infty\}/\sim,$$

where $\mathcal{M}[-\pi, \pi]$ denote the space of Lebesgue measurable complex valued functions on $[-\pi, \pi]$ and

$$f \sim g \Leftrightarrow f = g \;\; a.e. \text{ on } [-\pi, \pi].$$

1.6. **Fourier series as an eigenfunction expansion.** Let us look at the following *Dirac operator*

$$D := -i\frac{d}{dx} : f \mapsto -if',$$

on $C^\infty(S^1)$. It satisfies the following property.

Lemma 1.12. *$(Df, g) = (f, Dg)$ for every $f, g \in C^\infty(S^1)$.*

Proof. Recall that the inner product is defined by

$$(Df, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} -if'(x)\overline{g(x)}\, dx.$$

Thus the first formula follows directly from integration by parts

$$\int_{-\pi}^{\pi} f'(x)\overline{g(x)}\, dx = f(x)\overline{g(x)}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(x)\overline{g'(x)}\, dx = -\int_{-\pi}^{\pi} f(x)\overline{g'(x)}$$

where the second identity follows since f, g are 2π-periodic. □

Remark 1: In general, a linear operator T on $C^\infty(S^1)$ is said to be *self-adjoint* if

$$(Tf, g) = (f, Tg),$$

for every for every $f, g \in C^\infty(S^1)$. The above lemma implies that D is self-adjoint. Moreover, we know that the square of D is the *Laplacian operator* $\Box := -\frac{d^2}{dx^2}$, that is the reason why we call D the Dirac operator.

Remark 2: We know that *all Hermitian matrices are diagonalizable with real eigenvalues*. This fact is also true for our Dirac operator D, in fact we have

$$D(e^{inx}) = n(e^{inx}),$$

we call e^{inx} the *eigenfunction of* D *with eigenvalue* n. Thus we may look at the Fourier series expansion of $f \in C^\infty(S^1)$ as an *eigenfunction expansion* (compare it with the eigen-theory of matrices). Since $\{e^{inx}\}$ generates $f \in C^\infty(S^1)$, we know that D has no other eigenvalues.

Remark: Recall that is $s \in \mathbb{C}$ is not an eigenvalue of a matrix M then $M - s$ is invertible. Apply this fact to D, it is natural to ask whether $D + s$ is invertible in case $s \in \mathbb{C}\backslash\mathbb{Z}$. In fact, the Fourier series expansion defines the inverse of $D + s$ directly as follows

$$(D+s)^{-1}f : y \mapsto \sum \hat{f}(n)\frac{e^{iny}}{n+s}, \quad f \in C^\infty(S^1).$$

Since $\hat{f}(n) = \frac{-i}{n}\hat{f'}(n)$ and f is smooth we know that $\sum \hat{f}(n)\frac{e^{iny}}{n+s}$ also lies in $C^\infty(S^1)$. Moreover, $\hat{f}(n) = (f, e^{inx})$ gives

$$\left((D+s)^{-1}f\right)(y) = (f, G)\,,$$

where G is defined as follows:

Definition 1.13. *For every fixed* $s \in \mathbb{C}\backslash\mathbb{Z}$, $y \in (0, 2\pi)$, *we call*

$$G(x) := \sum_{n\in\mathbb{Z}} \frac{e^{in(x-y)}}{n+s}, \qquad x \in [0, 2\pi].$$

the y*-Green function of* $D + s$.

Recall that in *Example 2*, we proved that $\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}$ is the Fourier series expansion of

$$f(x) := \frac{\pi}{\sin \pi s} e^{i(\pi - x)s}, \qquad 0 \leqslant x \leqslant 2\pi.$$

Thus Theorem 1.5 gives

$$\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s} = \frac{\pi}{\sin \pi s} e^{i(\pi - x)s}, \ \ \forall\, x \in (0, 2\pi), \tag{4}$$

and the following crucial identity in *Eisenstein series* (see page 5 in [4])

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+s} = \frac{f(0) + f(2\pi)}{2} = \pi \cot \pi s, \tag{5}$$

i.e. $\sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n+s}$ is equal to a 2π-periodic function g such that

$$g(x) = \frac{\pi}{\sin \pi s} e^{i(\pi - x)s}, \ \ \forall\, x \in (0, 2\pi)$$

and

$$g(0) = \pi \cot \pi s,$$

which implies

Theorem 1.14. *We have* $G(y) = \pi \cot \pi s$,

$$G(x) = \frac{\pi}{\sin \pi s} e^{i(\pi - x + y)s}, \quad \forall\, x \in (y, 2\pi]$$

and

$$G(x) = \frac{\pi}{\sin \pi s} e^{-i(\pi + x - y)s}, \quad \forall\, x \in [0, y).$$

Proof. The first two formulas following directly (4) and (5). The last formula follows from

$$\sum_{n \in \mathbb{Z}} \frac{e^{inx}}{n+s} = g(x + 2\pi) = \frac{\pi}{\sin \pi s} e^{-i(\pi + x)s}, \ \ \forall\, x \in (-2\pi, 0).$$

□

By the above theorem, we know that G extends to a function in $PC^{\infty}(S^1)$, in particular, the $L^2(S^1)$ norm of G, say $||G||$, is finite, thus

$$G(x) = \sum_{n \in \mathbb{Z}} \frac{e^{-iny}}{n+s} \cdot e^{inx}$$

is the eigenfunction expansion (thus Fourier series expansion) of G and

$$\hat{G}(n) = \frac{e^{-iny}}{n+s}.$$

Remark: Another way of looking at G is the following: G is the unique 2π-*periodic distribution* on $\mathbb{R}$ that solves

$$(D + s)(G) = \sum_{k \in \mathbb{Z}} \delta_{y + 2\pi k},$$

where δ_ξ denotes the Dirac distribution (we will come back to it later) called *Dirac's delta function*.

1.7. **Some applications of Fourier series.**

1.7.1. *Wirtinger and Poincaré inequality.*

We shall follow page 91 in [9]. The first version of the Wirtinger inequality (also called optimal one-dimensional Poincaré inequality) is

Theorem 1.15. *Let* $f \in C^1(S^1)$ *with* $(f, 1) = 0$. *Then*

$$||f|| \leqslant ||f'||,$$

with equality if and only if $f(x) = f_1(x) = \hat{f}(1)e^{ix} + \hat{f}(-1)e^{-ix}$.

Proof. By the proof of Theorem 1.6, we have

$$\hat{f}(n) = \frac{-i}{n} \cdot \widehat{f'}(n).$$

Apply Bessel's inequality to f', we have

$$||f'||^2 \geqslant \sum_{|n| \leqslant N} |\widehat{f'}(n)|^2 = \sum_{|n| \leqslant N} n^2 |\hat{f}(n)|^2.$$

Since $(f, 1) = 0$ implies that $\hat{f}(0) = 0$, thus the Parseval's identity

$$||f||^2 = \lim_{N \to \infty} \sum_{0 < |n| \neq N} |\hat{f}(n)|^2 \leqslant ||f'||^2,$$

with inequality if and only if $\hat{f}(n) = 0$ for all $|n| > 1$, i.e. $f = f_1$. □

Remark: Notice that $|f'| = |Df|$, thus the above theorem gives

$$||Df|| \geqslant ||f||,$$

in case $f \in C^\infty(S^1)$ with $(f, 1) = 0$. The above identity is formally equivalent to that all non-zero eigenvalues, say λ, of D satisfy

$$|\lambda| \geqslant 1.$$

Theorem 1.15 also implies

Proposition 1.16. *Let $f \in C^1(S^1)$ with $(f, 1) = 0$. Then for every $g \in C^1(S^1)$, we have*

$$|(f, g)| \leqslant ||f|| \cdot ||g'||.$$

Proof. Notice that

$$(f, g) = (f, g - \hat{g}(0)).$$

Thus the theorem follows from Cauchy–Schwarz inequality and Theorem 1.15. □

The second version of the Wirtinger inequality

Theorem 1.17. *Let f be a C^1 function in a neighborhood of $[0, \pi]$ such that $f(0) = f(\pi) = 0$. Then*

$$\int_0^\pi |f(x)|^2\, dx \leqslant \int_0^\pi |f'(x)|^2\, dx,$$

with equality if and only if $f(x) = A \sin x$.

Proof. Check (try!) that $f|_{[0,\pi]}$ extends to an *odd* function (still denote it by f) in $C^1(S^1)$, in particular, $(f, 1) = 0$, thus Theorem 1.15 applies. □

Remark: The above proof of the Wirtinger inequality also applies to the famous *isoperimetric inequality* (see page 103 in [9]). A natural high diemsional generalization of the *convex version of the isoperimetric inequality* is the classical Brunn–Minkowski inequality (see [7]).

Theorem 1.18 (Brunn–Minkowski Theorem)**.** *Let A_0, A_1 be two compact convex sets in $\mathbb{R}^n$ with non-empty interior. Then*

$$|A_0 + A_1|^{\frac{1}{n}} \geqslant |A_0|^{\frac{1}{n}} + |A_1|^{\frac{1}{n}},$$

where $|A|$ denotes the Lebesgue measure (volume) of A and

$$A_0 + A_1 := \{x + y \in \mathbb{R}^n : x \in A_0,\ y \in A_1\}$$

is called the Minkowski sum of A_0 and A_1.

Remark: The above Brunn–Minkowski inequality is also true for every non-empty compact sets A_0 and A_1 (this general version is proved by Lazar Lyusternik in 1935), which can be seen as a generalization of the usual isoperimetric inequality.

1.7.2. *Heat equation on the circle.* We shall follow page 61–64 in [3].

The derivation of the heat equation is based on Newton's law of cooling, which states that the *flux* of heat across a point x_0 is proportional to the gradient of the temperature at x_0 (this is an experimental fact that is well verified for moderate temperature gradients, the full experimental relationship between flux and gradient is very complicated). This means that the amount of heat that flows past x_0 from left to right in a short time $[t_0, t_0 + \delta t]$ is approximately

$$-c_1 \cdot u_x(x_0, t_0) \cdot \delta t$$

with a positive constant c_1 depending upon the meterial (the minus sign is present because *heat flows from the hotter place to the cooler*). To proceed, the net amount of heat flowing *out* of a small interval $[x_0 - \delta x, x_0 + \delta x]$ in a short time $[t_0, t_0 + \delta t]$ is therefore

$$(-c_1 \cdot u_x(x_0 + \delta x, t_0) \cdot \delta t) - (-c_1 \cdot u_x(x_0 - \delta x, t_0) \cdot \delta t),$$

or approximately so. This can be computed in a second way: It is, in fact, proportional to the product of the length of the interval and the (average) decrease of the temperature inside. The constant of proportionality is the "specific heat" of the conducting material. Therefore,

$$-c_2 u_t(x_0, t_0) 2\delta x \cdot \delta t = (-c_1 \cdot u_x(x_0 + \delta x, t_0) \cdot \delta t) - (-c_1 \cdot u_x(x_0 - \delta x, t_0) \cdot \delta t),$$

or approximately so. Letting δx and δt go to zero, you find

$$u_t(x_0, t_0) = \frac{c_1}{c_2} u_{xx}(x_0, t_0).$$

We can replace the time coordinate t by $t = cT$, where c is another positive constant. If we set $U(x, T) = u(x, cT)$, then

$$U_T(x, T) = c u_t(x, cT) = \frac{cc_1}{c_2} u_{xx}(x, cT) = \frac{cc_1}{c_2} U_{xx}(x, T).$$

Choose c such that $2cc_1 = c_2$, it is enough to study the following standard heat equation

$$u_t = \frac{1}{2} u_{xx}.$$

In this section, we shall study the above heat equation on the circle $\mathbb{R}/\mathbb{Z}$ (of length one, i.e. the temperature is one-periodic with respect to x). Because u is supposed to be the temperature, it is natural to conjecture that the whole solution is determined by the temperature

$$f(x) := u(x, 0).$$

Since u is one-periodic with respect to x, we can formally write

$$u(x, t) = \sum_{n \in \mathbb{Z}} c_n(t) e^{2in\pi x},$$

with

$$c_n(t) = \int_0^1 u(x, t) e^{-2in\pi x} \, dx.$$

It is enough to compute $c_n(t)$

Lemma 1.19. *Each* $c_n(t)$ *satisfies*

$$c_n(0) = \int_0^1 f(x) e^{-2in\pi x} \, dx$$

and

$$c_n'(t) = -2\pi^2 n^2 c_n(t).$$

Proof. The first identity is just the initial condition. For the second identity, notice that

$$c_n'(t) = \int_0^1 u_t(x,t)e^{-2in\pi x}\,dx = \frac{1}{2}\int_0^1 u_{xx}(x,t)e^{-2in\pi x}\,dx.$$

Integration by parts gives

$$\int_0^1 u_{xx}(x,t)e^{-2in\pi x}\,dx = \int_0^1 u(x,t)(e^{-2in\pi x})_{xx}\,dx = -4\pi^2 n^2\int_0^1 u(x,t)e^{-2in\pi x}\,dx.$$

Thus the lemma follows. □

Solving the above ODE gives

$$c_n(t) = \left(\int_0^1 f(y)e^{-2in\pi y}\,dy\right)e^{-2\pi^2 n^2 t},$$

thus

$$u(x,t) = \sum\left(\int_0^1 f(y)e^{-2in\pi y}\,dy\right)e^{-2\pi^2 n^2 t}e^{2in\pi x} = \int_0^1 \theta(x-y, 2\pi i t)\,f(y)\,dy$$

where θ denotes the classical Riemann theta function defined by

$$\theta(z,t) := \sum_{n\in\mathbb{Z}} e^{\pi i(n^2 t + 2nz)},\ \ z\in\mathbb{C},\ t\in\mathbb{H} := \{t\in\mathbb{C} : \operatorname{Im} t > 0\}. \tag{6}$$

One may check that (try!) θ is holomorphic on $\mathbb{C}\times\mathbb{H}$.

Definition 1.20. *We call θ the Jacobi theta function and*

$$h(x,y,t) := \theta(x-y, 2\pi i t),\ \ x,y\in\mathbb{R},\ t>0,$$

the heat kernel on the circle.

The heat kernel above solves the heat equation in the following sense

Theorem 1.21. *For $f\in C^2(\mathbb{R}/\mathbb{Z})$, put*

$$u(x,t) = \int_0^1 h(x,y,t)\,f(y)\,dy.$$

Then we have

1) *u is one-periodic with respect x and is smooth on $\mathbb{R}\times(0,\infty)$;*
2) *$u_t = \frac{1}{2}u_{xx}$ on $\mathbb{R}\times(0,\infty)$;*
3) $\lim_{t\to 0+}\sup_{x\in\mathbb{R}}|u(x,t)-f(x)| = 0.$

Proof. It is easy to check 1) and 2) since $e^{-2\pi^2 n^2 t}$ is a rapidly decreasing function of n. 3) follows from

$$|u(x,t)-f(x)| \leqslant \sum_{n\in\mathbb{Z}}(1-e^{-2\pi^2 n^2 t})|\hat f(n)|,$$

and

$$|\hat f(n)| = |(2\pi i n)^{-2}\widehat{f''}(n)| \leqslant (2\pi n)^{-2}\int_0^1 |f''(x)|\,dx.$$

□

Remark: The above theorem is also true for $f\in C^0(\mathbb{R}/\mathbb{Z})$, for the proof and the uniqueness of u, see page 64–65 in [3].

1.7.3. *Weyl's equidistribution theorem.* We shall follow page 106–112 in [9], for related results, see page 54–56 in [3]. A basic postulate of statistical mechanics is the so called *ergodic principle* of Boltzman and Gibbs, which states that the time average of a mechanical quantity should be the same as its phase average; see Ford and Uhlenbeck [6] (page 9–13) for a nice discussion of such matters. A simple instance of this phenomenon can be seen in the following model due to Weyl in 1916.

As *phase space*, bring in the circle $\mathbb{R}/\mathbb{Z}$, pick a number $0 < \gamma < 1$, and look at the *rotation*

$$x \mapsto x_1 := x + \gamma$$

with addition modulo 1. The *trajectory* of the phase point $x_0 = x$ is the arithmetic sequence

$$x_0 = x,\ x_1 = x + \gamma,\ \cdots,\ x_n = x + n\gamma,\ \cdots,$$

considered modulo 1. A *mechanical quantity* is a function $f \in PC^0(\mathbb{R}/\mathbb{Z})$. Its *time average* is

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(x_k)}{n},$$

assuming this limit to exist, while its *phase everage* is just the arithmetic mean

$$\int_0^1 f(x)\,dx.$$

Weyl proved that

Theorem 1.22. *If γ is irrational then*

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(k\gamma)}{n} = \int_0^1 f(x)\,dx,$$

for every $f \in C^0(\mathbb{R}/\mathbb{Z})$.

Proof. Use Fejér's theorem to approximate f uniformaly by trigonometric polynomials (see Exercise set 1), it is enough to prove the theorem for $f = e^{2\pi i m x}$. If $m = 0$ then both sides are 1. If $m \neq 0$ then

$$\sum_{k=0}^{n-1} f(k\gamma) = \sum_{k=0}^{n-1} a^k,\ \ a := e^{2\pi i m\gamma}.$$

Since γ is irrational, we know that $a \neq 1$ and

$$\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$$

is bounded by $|1-a|^{-1}$. Thus

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(k\gamma)}{n} = 0 = \int_0^1 f(x)\,dx.$$

The proof is complete. □

Remark: Fix two real numbers $a < b$ with $b - a < 1$, consider the following one-periodic indicator function defined by

$$1_{[a,b]+\mathbb{Z}}(x) := 1 \text{ if } x + n \in [a,b],$$

for some ineteger n and $1_{[a,b]+\mathbb{Z}}(x) := 0$ otherwise. One may prove that the above theorem also applies to $1_{[a,b]+\mathbb{Z}}$: the idea is to approximate $1_{[a,b]+\mathbb{Z}}$ above and below by continuous functions f^+

and f^- so as to make $\int_0^1 (f^+ - f^-)\, dx$ small and use the above theorem to f^+ and f^- respectively. Thus we get: if γ is irrational then

$$\lim_{n\to\infty} \frac{\#\{k < n : k\gamma \in [a,b] + \mathbb{Z}\}}{n} = b - a,$$

and we say that $\{k\gamma + \mathbb{Z}\}$ is *equidistributed* in $\mathbb{R}/\mathbb{Z}$. A generalization of this fact is the following Weyl's criterion

Theorem 1.23. *Let $\{\xi_n\}_{n=0}^{\infty}$ be a sequence of real number. Then $\{\xi_n + \mathbb{Z}\}$ is equidistributed in $\mathbb{R}/\mathbb{Z}$ if and only if for every nonzero integer k,*

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} e^{2\pi i k \xi_n}}{N} = 0.$$

Proof. $\Rightarrow$: Since equidistributive property of $\{\xi_n + \mathbb{Z}\}$ is equivalent to that for every one-periodic indicator function f,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} f(\xi_n)}{n} = \int_0^1 f(x)\, dx.$$

Approximating $f \in C^0(S^1)$ by one-periodic indicator function, we know that the above identity is also true for every $f \in C^0(S^1)$. Now it is enough to apply it to $f(x) = e^{2\pi i k x}$.

$\Leftarrow$: Follows by a similar argument as in the proof of equidistributive property of $\{k\gamma + \mathbb{Z}\}$. □

1.8. **Several dimensional Fourier series.** We shall follow page 81–85 in [3].

Definition 1.24. *A function f on $\mathbb{R}^d$ is said to be $\mathbb{Z}^d$-invariant if*

$$f(x+k) = f(x), \qquad \forall\, k \in \mathbb{Z}^d.$$

We call a $\mathbb{Z}^d$-invariant function a function on the standard torus T^d.

We shall denote by $L^2(T^d)$ the completion of the space $C^\infty(T^d)$ of smooth $\mathbb{Z}^d$-invariant functions with respect to the following inner product

$$(f,g) := \int_0^1 \cdots \left(\int_0^1 f(x_1, \cdots, x_d)\overline{g(x_1, \cdots, x_d)}\, dx_1 \right) dx_2 \cdots dx_d, \ \ f, g \in C^\infty(T^d).$$

Since finite $\mathbb{C}$-linear combinations of functions in $\{1|_Q\}$, where Q denotes an arbitray n-cube, are dense in $L^2(T^d)$ and

$$1|_Q(x) = 1|_{Q_1}(x_1) \cdots 1|_{Q_d}(x_d), \qquad Q := Q_1 \times \cdots \times Q_d,$$

we know that

$$e_Z(x) := e^{2\pi i Z\cdot x}, \ \ Z \in \mathbb{Z}^d$$

defines an orthonormal basis of $L^2(T^d)$, which gives

1.8.1. *Fourier series on a standard Torus.*

Theorem 1.25. *Every $f \in L^2(T^d)$ satisfies*

$$\lim_{N\to\infty} ||f - \sum_{Z\in[-N,N]^d} \hat{f}(Z)\, e_Z|| \to 0,$$

(later we shall write the above identity as $f = \sum_{Z\in\mathbb{Z}^d} \hat{f}(Z) e_Z$) where

$$\hat{f}(Z) := (f, e_Z),$$

denotes the Z-th Fourier coefficient of f.

1.8.2. *Application to Random walks.* Pólya discovered a very beautiful application of several dimensional Fourier series to "random walks". Think of a particle moving on the d-dimensional lattice $\mathbb{Z}^d$ according to the following rule. The particle starts at time 0 at the origin and moves at time $n \geqslant 1$ by a unit step u_n to a neighborhood lattice point; for example, if $d = 3$, the possible steps are

$$u = (\pm 1, 0, 0), \quad (0, \pm, 0), \quad (0, 0, \pm 1), \qquad |u| = 1.$$

The position of the particle at time $n \geqslant 1$ is the sum of the individual steps: $s_n = u_1 + \cdots + u_n$. The step u_n is statistically independent of the preceding steps $u_j : j < n$ and the possible steps are equally likey at each stage. This means that

$$P(u_1 = \hat{u}_1, \cdots, u_n = \hat{u}_n) = P(u_1 = \hat{u}_1) \times \cdots \times P(u_n = \hat{u}_n) = (2d)^{-n}$$

for any fixed unit steps $\hat{u}_1, \cdots, \hat{u}_n$, in which $P(E)$ means "the probability of the event E".

The problem is to compute $P(s_n = Z)$ and to study the behavior of s_n for $n \to \infty$.

Pólya's idea is to think of $P(s_n = Z)$ as the Fourier coefficient $\hat{f}(Z)$ of a function $f \in L^2(T^n)$:

$$f(x) = \sum P(s_n = Z) e_Z(x) = \sum_{Z \in \mathbb{Z}^d} P(s_n = Z) e^{2\pi i Z \cdot x}.$$

The sum is just the "expectation" or "mean value" of $e^{2\pi i s_n \cdot x}$ and is easily computed using the independence of the individual steps. In fact, notice that

$$P(s_n = Z) = \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} P(u_1 = \hat{u}_1) \times \cdots \times P(u_n = \hat{u}_n) = \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} (2d)^{-n}.$$

Thus

$$P(s_n = Z) e^{2\pi i Z \cdot x} = \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} (2d)^{-n} e^{2\pi i \, (\hat{u}_1 + \cdots + \hat{u}_n) \cdot x},$$

which gives

$$f(x) = \sum_{Z \in \mathbb{Z}^d} \sum_{\hat{u}_1 + \cdots + \hat{u}_n = Z} (2d)^{-n} e^{2\pi i \, (\hat{u}_1 + \cdots + \hat{u}_n) \cdot x} = \left((2d)^{-1} \sum_{|u|=1} e^{2\pi i u \cdot x} \right)^n.$$

Since

$$\sum_{|u|=1} e^{2\pi i u \cdot x} = 2 \left(\cos 2\pi x_1 + \cdots + \cos 2\pi x_d \right),$$

put

$$f_d(x) := \frac{\cos 2\pi x_1 + \cdots + \cos 2\pi x_d}{d},$$

we get

$$f(x) = f_d(x)^n.$$

Thus Theorem 1.25 gives the following *Pólya's formula*

$$P(s_n = Z) = \hat{f}(Z) = (f_d^n, e_Z).$$

In particular,

$$P(s_n = 0) = \int_{[0,1]^d} f_d(x)^n \, dx_1 \cdots dx_d.$$

Since $|f_d| \leqslant 1$, the expected number of times the particle visits the origin can be expressed as

$$\sum_{n=0}^{\infty} P(s_n = 0) = \lim_{\varepsilon \to 1} \sum_{n=0}^{\infty} \varepsilon^n P(s_n = 0) = \lim_{\varepsilon \to 1} \int_{[0,1]^d} \sum_{n=0}^{\infty} \varepsilon^n f_d(x)^n \, dx_1 \cdots dx_d.$$

Since

$$\lim_{\varepsilon\to 1}\int_{[0,1]^d}\sum_{n=0}^{\infty}\varepsilon^n f_d(x)^n = \lim_{\varepsilon\to 1}\int_{[0,1]^d}(1-\varepsilon f_d)^{-1} = \int_{[0,1]^d}(1-f_d)^{-1},$$

we get

$$\sum_{n=0}^{\infty} P(s_n = 0) = \int_{[0,1]^d}\frac{1}{1-\frac{\cos 2\pi x_1+\cdots+\cos 2\pi x_d}{d}}\,dx_1\cdots dx_d.$$

By the definition of e^x in Appendix 1 and the Euler formula, we get

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots,$$

which implies that

$$\frac{x^2}{3} \leqslant 1-\cos x \leqslant \frac{x^2}{2},$$

when $|x|$ is small enough. Thus (leave as an exercise) $\sum_{n=0}^{\infty} P(s_n = 0) < \infty$ if and only if

$$\int_{[0,\varepsilon]^d}\frac{1}{x_1^2+\cdots+x_d^2}\,dx_1\cdots dx_d < \infty,$$

which is equivalent to $d \geqslant 3$ (by using the polar coordinate). Pólya used this to prove a very striking fact about the ultimate begavior of the walk.

Theorem 1.26. *If $d \geqslant 3$ then*

$$P(\lim_{n\to\infty}|s_n| = \infty) = 1.$$

If $d \leqslant 2$ then

$$P(s_n = 0 \textit{ infinitely often}) = 1.$$

Proof. If $d \geqslant 3$ then we know that the expected number, say P, of times the particle visits the origin is less than infinity. Denote by p_n the probability of n actural number of visits. We know that

$$P = p_\infty \times \infty + \sum_{n=0}^{\infty} np_n < \infty, \;\; p_\infty + \sum_{n=0}^{\infty} p_n = 1.$$

Thus we must have $p_\infty = 0$ and the actual number of visits is less than infinity with probability 1, and since the origin is not special in any way, the same must be true for every lattice point in $\mathbb{Z}^d$. This means that for any $R < \infty$, the particle ultimately stops visiting the ball $|Z| < R$, and that is the same as to say

$$P(\lim_{n\to\infty}|s_n| = \infty) = 1.$$

Now let us assume that $d \leqslant 2$. At time $n = 1$, the particle steps to one of the $2d$ nearest neighbors of the origin. The problem is to check that the probability p of ultimately returning to the origin is 1. In fact, the probability of visiting the origin m or more times (including the visit at time $n = 0$) is p^{m-1}. Thus the probability of precisely m visits is

$$p^{m-1} - p^m = p^{m-1}(1-p).$$

If $p < 1$ then $p^\infty = 0$ and the expected number of visits would be

$$\infty \times 0 + \sum_{m=1}^{\infty} mp^{m-1}(1-p) = (1-p)^{-1} < \infty,$$

contradicting the evaluation that $\sum_{n=0}^{\infty} P(s_n = 0) = \infty$. The proof is finished; for additional information on the subject, see Feller [5], pp 342–371. □

1.8.3. *Fourier series on a two dimensional Torus.* Pick numbers $a \in \mathbb{R}$, $b > 0$ and introduce the "non-standard" lattice $L \subset \mathbb{R}^2$ of all points of the form

$$\omega = j(1,0) + k(a,b), \qquad (j,k) \in \mathbb{Z}^2.$$

Definition 1.27. *A function f on $\mathbb{R}^2$ is said to be L-periodic if*

$$f(x + (1,0)) = f(x) = f(x + (a,b)),$$

for every $x \in \mathbb{R}^2$.

Definition 1.28. *We call the set of all points $\omega' \in \mathbb{R}^2$ such that*

$$\omega' \cdot \omega \in \mathbb{Z},$$

for every $\omega \in L$ the dual lattice of L, and denote it by L'.

Exercise: Check that L' is the lattice of points

$$\omega' = j(1, -\frac{a}{b}) + k(0, \frac{1}{b}), \qquad (j,k) \in \mathbb{Z}^2.$$

$L' = L$ if and only if $a \in \mathbb{Z}$ and $b = 1$.

Exercise: Check that

$$e_\gamma(x) := e^{2\pi i \gamma \cdot x}$$

is L-periodic if and only if $\gamma \in L'$.

Remark: One may look at the torus

$$T_L := \mathbb{R}^2 / L$$

by identifying opposite sides of the following "fundamental cell"

$$F_L := \{t(1,0) + s(a,b) : 0 \leqslant t, s \leqslant 1\}.$$

Denote by $C^\infty(T_L)$ the space of smooth L-periodic functions on $\mathbb{R}^2$. Let $L^2(T_L)$ be the completion of $C^\infty(T_L)$ with respect to the following inner product

$$(f,g)_L := \int_{F_L} f(x)\overline{g(x)}\, dx_1 \cdots dx_n, \qquad f, g \in C^\infty(T_L).$$

Then we have the following generalization of the standard torus Fourier series expansion.

Theorem 1.29. *Every $f \in L^2(T_L)$ has the following orthogonal decomposition*

$$f = \sum_{\gamma \in L'} \hat{f}(\gamma) e_\gamma,$$

where

$$\hat{f}(\gamma) := (f, e_\gamma)_L.$$

Moreover, the following Plancherel identity holds

$$||f||_L^2 = b \cdot \sum_{\gamma \in L'} |\hat{f}(\gamma)|^2.$$

Proof. Think of f and e_γ as functions of $y_1 = x_1 - \frac{a}{b} x_2$ and $y_2 = \frac{1}{b} x_2$. This will bring you back to the standard torus case. □

Exercise: Try to write down the details of the proof of the above theorem (notice that b is the area of F_L; if f is Z-periodic then

$$g(y) := f(y_1 + ay_2, by_2),$$

is $\mathbb{Z}^n$-invariant).

Exercise: Try to develop similar Fourier series theory for general high dimensional torus.

2. Fourier transform

Recall that if a smooth function on $\mathbb{R}$ is $2L$-periodic then the following Fourier series expansion holds

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2L} \int_{-L}^{L} f(y) e^{-2\pi i n \frac{y}{2L}} \, dy \right) e^{2\pi i n \frac{x}{2L}}.$$

A small change in viewpoint leads at once to the Fourier integral: the idea is that the right hand side is really a Riemann sum over a subdivision with spacing $\frac{1}{2L}$, and with any luck, it should approximate the integral

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-2\pi i y\gamma} \, dy \right) e^{2\pi i x\gamma} \, d\gamma$$

as $L \to infty$. This does not make too much sense for a periodic function f (the integral cannot converge well), but it does suggest that something can be done to recover a nice function f from its *Fourier integral (or transform)*:

$$\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(y) e^{-2\pi i y\gamma} \, dy$$

via the *inverse Fourier integral*

$$\check{\hat{f}} = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i x\gamma} \, d\gamma.$$

The purpose of the next two sections is put this formal discussion on a solid mathematical foundation.

2.1. **Fourier transform on the Schwartz space.**

Definition 2.1. *By the Schwartz space, say $\mathcal{S}$, on $\mathbb{R}$, we mean the space of all smooth functions, say f, on $\mathbb{R}$ such that*

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty,$$

for every non-negative integers k, l, where $f^{(l)}$ denotes the l-th order derivative of f.

Examples: $e^{-x^2} \in \mathcal{S}$ but $(1+x^2)^{-1}$ does not belong to $\mathcal{S}$; nor does $e^{-|x|}$, but for a different reason (try to verify this statement).

Definition 2.2. *The Fourier transform of a function $f \in \mathcal{S}$ is defined by*

$$\hat{f}(\gamma) := \int_{-\infty}^{\infty} f(y) e^{-2\pi i y\gamma} \, dy.$$

The inverse Fourier transform of a function $g \in \mathcal{S}$ is defined by

$$\check{g}(x) = \int_{-\infty}^{\infty} g(\gamma) e^{2\pi i x\gamma} \, d\gamma.$$

Remark: We have $\hat{f}(x) = \check{f}(-x)$.

Proposition 2.3. *If $f \in \mathcal{S}$ then*

1) $\widehat{f(x+h)} = \hat{f}(\gamma)e^{2\pi i h\gamma}$ *whenever* $h \in \mathbb{R}$*;*
2) $\widehat{f(x)e^{-2\pi i h x}} = \hat{f}(\gamma + h)$ *whenever* $h \in \mathbb{R}$*;*
3) $\widehat{f(\delta x)} = \delta^{-1}\hat{f}(\delta^{-1}\gamma)$ *whenever* $\delta > 0$*;*
4) $\widehat{f'(x)} = 2\pi i\gamma\hat{f}(\gamma)$*;*
5) $\widehat{-2\pi i x f}(x) = \frac{d}{d\gamma}\hat{f}(\gamma)$.

Proof. We only prove 3) and leave the others as exercises (see page 136–137 in [9]). Integration by parts gives

$$\int_{-N}^{N} f'(x)e^{-2\pi i x\gamma}\,dx = f(x)e^{-2\pi i x\gamma}|_{-N}^{N} + 2\pi i\gamma\int_{-N}^{N} f(x)e^{-2\pi i x\gamma}\,dx,$$

so letting N goes to infinity gives 3). □

Example: Show that

$$\widehat{e^{-\pi x^2}} = e^{-\pi\gamma^2},$$

which implies that for every $s > 0$,

$$\widehat{e^{-\pi x^2 s}} = s^{-\frac{1}{2}}e^{-\pi\gamma^2/s}. \tag{7}$$

Remark: By 4) and 5), we know that the Fourier transform interchanges differentiation and multiplication, which can be used to prove (try!) the following result.

Theorem 2.4. *If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$.*

Another key fact is that the Fourier transform interchanges convolutions with pointwise products.

Definition 2.5. *For every $f, g \in \mathcal{S}$, we call*

$$f \star g : x \mapsto \int_{-\infty}^{\infty} f(x-y)g(y)\,dy,$$

the convolution of f and g.

Proposition 2.6. *If $f, g, h \in \mathcal{S}$ then*

1) $f \star g \in \mathcal{S}$*;*
2) $f \star g = g \star f$*;*
3) $\widehat{f \star g} = \hat{f}\hat{g}$*;*
4) $(f \star g) \star h = f \star (g \star h)$*;*
5) $(f \star g)' = f' \star g = g' \star f$.

Proof. Exercise (see page 142–143 in [9]). □

2.2. **Classical Poisson summation formula.** We need a lemma to state the classical Poisson summation formula.

Lemma 2.7. *Let f be a continuous function on $\mathbb{R}$ such that*

$$|f(x)| \leqslant C(1+x^2)^{-1}.$$

Then

$$f_T(x) := \sum_{k\in\mathbb{Z}} f(x+kT),$$

defines a T-periodic continuous function on $\mathbb{R}$ such that

$$|f_T(x) - f(x)| \leqslant C\frac{\pi^2}{T^2}, \ \forall \ |x| \leqslant \frac{T}{2}.$$

Proof. Put

$$f_N(x) := \sum_{|k|\leqslant N} f(x+kT).$$

Then we know that both f_N and f'_N converges uniformly to f_T. Thus f_T is C^1 and obviously f_T is T-periodic. The final estimate follows from

$$|f_T(x) - f(x)| \leqslant \frac{2C}{T^2}\sum_{k=1}^{\infty}\frac{1}{(k-\frac{1}{2})^2} = C\frac{\pi^2}{T^2}.$$

□

Remark: If $f \in \mathcal{S}$ then every n-th derivative $f^{(n)}$ of f fits the above lemma, thus f_T is smooth and T-periodic. Apply the Fourier series expansion, we get

$$f_T(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n \frac{x}{T}},$$

where

$$c_n = T^{-1}\int_{|y|<\frac{T}{2}} f_T(y) e^{-2\pi i n\frac{y}{T}}\, dy.$$

Notice that

$$\int_{|y|<\frac{T}{2}} f_T(y)e^{-2\pi i n\frac{y}{T}}\, dy = \sum_{k\in\mathbb{Z}}\int_{|y|<\frac{T}{2}} f(y+kT)e^{-2\pi i n\frac{y}{T}}\, dy = \int_{\mathbb{R}} f(y)e^{-2\pi i n\frac{y}{T}}\, dy = \hat{f}\left(\frac{n}{T}\right),$$

which gives

Theorem 2.8. *If $f \in \mathcal{S}$ then*

$$f_T(x) = \sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{n}{T}\right) e^{2\pi i n\frac{x}{T}}.$$

In particular, we have the following Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \tag{8}$$

Application to Jacobi theta identities: Apply the above theorem to

$$f(x) = e^{\pi i(x^2 t + 2xz)}, \ \ z, t \in \mathbb{C}, \ \operatorname{Im} t > 0,$$

by (7), we get

$$\hat{f}(\lambda) = (-it)^{-\frac{1}{2}} e^{-i\pi\frac{(\gamma - z)^2}{t}}.$$

Thus by the Poisson summation formula and the definition of theta function in (6), we get

$$\theta(z,t) = (-it)^{-\frac{1}{2}} e^{-\frac{i\pi z^2}{t}}\theta\left(\frac{z}{t}, -\frac{1}{t}\right). \tag{9}$$

The readers can easily check the remaining theta identities

$$\theta(z, t+1) = \theta\left(z + \frac{1}{2}, t\right), \ \ \theta(z+1, t) = \theta(z,t), \ \ \theta(z,t) = e^{\pi i(t+2z)}\theta(z+t, t). \tag{10}$$

Remark: Notice that

$$f_T(x) = \sum_{n\in\mathbb{Z}} e^{\pi i\left((x+kT)^2 t+2(x+kT)z\right)} := \theta(z,t;x,T),$$

then the above theorem implies

$$\theta(z,t;x,T) = \sum_{n\in\mathbb{Z}} (-it)^{-\frac{1}{2}} e^{-i\pi\frac{\left(\frac{n}{T}-z\right)^2}{t}} e^{2\pi i n\frac{x}{T}} = (-it)^{-\frac{1}{2}} e^{-\frac{i\pi z^2}{t}} \theta\left(\frac{z}{t}+x, -\frac{1}{t}; 0, \frac{1}{T}\right). \tag{11}$$

2.3. **Fourier inversion formula and Plancherel identity.** The Poisson summation formula implies the following result (for a direct *Fourier series expansion* proof, see page 89 in [3]).

Theorem 2.9. *Every f in $\mathcal{S}$ satisfies the Fourier inversion formula*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\gamma) e^{2\pi i x\gamma}\, d\gamma$$

and the Plancherel identity

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\gamma)|^2\, d\gamma.$$

Proof. TBA, see □

Denote by

$$(f,g) := \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

the inner product space structure on $\mathcal{S}$. Put

$$||f|| := (f,f)^{\frac{1}{2}}.$$

Definition 2.10. *We call the set of all equivalent classes of Cauchy sequences in $\mathcal{S}$ with the above inner product the completion of $\mathcal{S}$ and denote it by $L^2(\mathbb{R})$.*

Remark: Anther way to look at $L^2(\mathbb{R})$ is to use the *Lebesgue integral* theory (see Appendix 2), which gives the following isomorphism

$$L^2(\mathbb{R}) \simeq \{f \in \mathcal{M}(\mathbb{R}) : \int_{-\infty}^{\infty} |f|^2 dx < \infty\}/\sim,$$

where $\mathcal{M}(\mathbb{R})$ denote the space of Lebesgue measurable complex valued functions on $\mathbb{R}$ and

$$f \sim g \Leftrightarrow f = g \ \ a.e. \text{ on } \mathbb{R}.$$

The Plancherel identity implies that both the Fourier transform $\hat{f}$ and the Fourier inversion $(\hat{f})^\vee$ extend to $L^2(\mathbb{R})$ on which the Fourier inversion formula and the Plancherel identity still hold. See *Exercise set 4* for a canonical basis of $L^2(\mathbb{R})$ using eigenfunctions of the Fourier transform.

2.4. **Fourier transform of tempered distributions.** In applications (e g elementary solution or Green's function of a partial differential operator, see Theorem 7.1.20 in [8]), it is crucial to extend Fourier transforms to a larger class of functions. A natural way to do it is to use the notion of distribution by Schwartz.

Test functions: Denote by $\mathcal{D}_R$ the space of smooth functions, say f, on $\mathbb{R}$ such that

$$f(x) = 0, \quad \text{if } |x| \geqslant R.$$

Put

$$\mathcal{D} = \cup_{R>0} \mathcal{D}_R.$$

We call $\mathcal{D}$ the space of *test functions*. It is clear that $\mathcal{D}$ is a subspace of $\mathcal{S}$.

Exercise: Check that the classical cut-off function

$$\chi(x) := c e^{\frac{1}{|x|-1}}, \ \ |x| < 1; \quad \chi(x) := 0, \ \ |x| \geqslant 1, \tag{12}$$

is smooth on $\mathbb{R}$, where c is choosing such that $\int_{\mathbb{R}} \chi \, dx = 1$. For $\varepsilon > 0$, put

$$\chi_\varepsilon(x) := \varepsilon^{-1} \chi(\varepsilon^{-1} x),$$

then $\chi_\varepsilon \in \mathcal{D}_\varepsilon$. Moreover, for every $\delta > \varepsilon$,

$$\chi_{\varepsilon,\delta}(x) := \int_{|y| \leqslant \delta} \chi_\varepsilon(x - y) \, dy$$

lies in $\mathcal{D}_{\varepsilon+\delta}$ and $\chi_{\varepsilon,\delta}(x) = 1$ if $|x| \leqslant \delta - \varepsilon$.

Definition 2.11. *A $\mathbb{C}$-linear map*

$$T : \mathcal{D} \to \mathbb{C}$$

is said to be a distribution on $\mathbb{R}$ if for every $R > 0$ there exists a positive constant $C(R)$ and a positive integer $N(R)$ such that

$$|T(f)| \leqslant C(R) \sup_{|x|<R, \, 0 \leqslant n \leqslant N(R)} |f^{(n)}(x)|, \qquad \forall \, f \in \mathcal{D}_R.$$

A distribution T on $\mathbb{R}$ is said to be tempered if it extends to a $\mathbb{C}$-linear map, still denote it by T,

$$T : \mathcal{S} \to \mathbb{C},$$

such that there exists a positive constant C and positive integers N and M with

$$|T(f)| \leqslant C P_{N,M}(f), \ \ P_{N,M}(f) := \sup_{x \in \mathbb{R}, \, 0 \leqslant k \leqslant N, \, 0 \leqslant l \leqslant M} |x|^k |f^{(l)}(x)|, \qquad \forall \, f \in \mathcal{S}.$$

We shall denote by $\mathcal{D}'$ the space of all distrubutions on $\mathbb{R}$ and by $\mathcal{S}'$ the space of all tempered distributions on $\mathbb{R}$.

Remark: Notice that (try!)

$$\lim_{n \to \infty} P_{N,M}(f - \chi_{1,n} f) \to 0 = 0, \qquad \forall \, f \in \mathcal{S}$$

implies that every tempered distribution is uniquely determined by its restriction on $\mathcal{D}$.

Examples of distribution:

Piecewise continuous functions: If f is a piecewise continuous function then

$$T_f : g \mapsto \int_{\mathbb{R}} f(x) g(x) \, dx,$$

defines a distribution (sometimes we shall identity f with T_f). Check that T_{e^x} is not tempered.

Dirac's delta function: Fix $\xi \in \mathbb{R}$, the Delta funcion

$$\delta_\xi : f \mapsto f(\xi),$$

defines a distribution on $\mathbb{R}$, moreover $\delta_\xi \in \mathcal{S}'$

L^2*-functions*: One may look at $L^2(\mathbb{R})$ as a subspace of $\mathcal{S}'$: in fact, for every $f = [\{f_n\}]$,

$$T_f : g \mapsto \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\,dx, \qquad g \in \mathcal{S},$$

defines a tempered distribution (we will identify f with T_f) since

$$\Big|\int_{\mathbb{R}} f_n(x)g(x)\,dx\Big| \leqslant \int_{\mathbb{R}} |f_n(x)|(1+x^2)^{-\frac{1}{2}}\,dx \cdot \sup_{x\in\mathbb{R}} |(1+x^2)^{\frac{1}{2}}|g(x)|$$

and

$$\int_{\mathbb{R}} |f_n(x)|(1+x^2)^{-\frac{1}{2}}\,dx \leqslant ||f_n|| \left(\int_{\mathbb{R}} (1+x^2)^{-1}\,dx\right)^{\frac{1}{2}} = \pi^{\frac{1}{2}} \cdot ||f_n||$$

give

$$|T_f(g)| \leqslant (2\pi)^{\frac{1}{2}} \cdot ||f|| \sup_{x\in\mathbb{R},\ 0\leqslant k\leqslant 1} |x|^k|g(x)|.$$

Cauchy principal values: The Cauchy principal value

$$p.v.\left(\frac{1}{x}\right) : f \mapsto \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

defines a tempered distribution. In fact, we have

$$\int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx = \int_{\varepsilon}^{\infty} \frac{f(x)-f(-x)}{x}\,dx.$$

Since

$$\Big|\frac{f(x)-f(-x)}{x}\Big| = \Big|\frac{1}{x}\int_{-x}^{x} f'(t)\,dt\Big| \leqslant 2 \sup_{|t|\leqslant|x|} |f'(t)|,$$

we get

$$\Big|\int_{\varepsilon}^{1} \frac{f(x)-f(-x)}{x}\,dx\Big| \leqslant 2\sup_{x\in\mathbb{R}} |f'(x)|,$$

and

$$\Big|\int_{1}^{\infty} \frac{f(x)-f(-x)}{x}\,dx\Big| \leqslant 2\sup_{x\in\mathbb{R}} |xf(x)| \int_{1}^{\infty} \frac{1}{x^2}\,dx = 2\sup_{x\in\mathbb{R}} |xf(x)|.$$

Thus

$$\Big|\Big[p.v.\left(\frac{1}{x}\right)\Big](f)\Big| \leqslant 2\sup_{x\in\mathbb{R}} |f'(x)| + 2\sup_{x\in\mathbb{R}} |xf(x)|,$$

which implies that $p.v.\left(\frac{1}{x}\right)$ defines a tempered distribution.

Derivative of a distribution. Let $T \in \mathcal{D}'$. Then derivatives of T can always be defined

$$T^{(k)} : f \mapsto T((-1)^k f^{(k)}).$$

Notice that $T \in \mathcal{S}'$ implies that $T^{(k)} \in \mathcal{S}'$. If g is a smooth function then $T_g^{(k)} = T_{g^{(k)}}$.

Multiplication by smooth functions: Let $f \in \mathcal{D}, T \in \mathcal{D}'$. Then

$$fT(g) := T(fg),$$

defines $fT \in \mathcal{D}'$. It is easy to check that

$$xT \in \mathcal{S}', \quad fT \in S',$$

if $f \in \mathcal{S}, T \in \mathcal{S}'$.

2.4.1. *Fourier transform of a tempered distribution.* The motivation for extending Fourier transform to tempered distributions comes from the following identity (which is sometimes called the multiplication formula)

Theorem 2.12. *If $f, g \in \mathcal{S}$ then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)\,dy.$$

Proof. Put $F(x,y) = f(x)g(y)e^{-2\pi i x y}$ then the theorem follows from

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\,dy\right)dx = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} F(x,y)\,dx\right)dy,$$

the details are left as an exercise. □

Definition 2.13 (Fourier transform on $\mathcal{S}'$)**.** *Let $T \in \mathcal{S}'$, put*

$$\hat{T} : f \mapsto T(\hat{f}), \quad \check{T} : f \mapsto T(\check{f}),$$

we call $\hat{T}$ the Fourier transform of T and $\check{T}$ the inverse Fourier transform of T.

Remark: The following lemma implies that $\hat{T}, \check{T} \in \mathcal{S}'$ if $T \in \mathcal{S}'$.

Lemma 2.14. *For every $f \in \mathcal{S}$, we have*

$$P_{N,M}(\hat{f}) \leqslant C(M,N)P(M+2,N)(f)$$

Proof. Follows from

$$|x^k \hat{f}^{(l)}(x)| = |\widehat{(y^l f)^{(k)}}(x)| \leqslant \int_{\mathbb{R}} |(y^l f)^{(k)}|\,dy \leqslant \pi \sup_{y\in\mathbb{R}}(1+y^2)|(y^l f)^{(k)}|.$$

□

One may define $f \star T$ ($f \in \mathcal{S}, T \in \mathcal{S}'$) as follows

$$(f \star T)(h) := T(f^- \star h), \qquad h \in \mathcal{S},$$

where $f^-(x) := f(-x)$. Then the following basic properties of the Fourier transform can be naturally generalized to tempered distributions.

Theorem 2.15. *For every $f, g \in \mathcal{S}$, $T \in \mathcal{S}'$, we have*

1) $\widehat{f \star T} = \hat{f}\hat{T}$;
2) $\widehat{T^{(k)}} = (2\pi i x)^k \hat{T}$;
3) $\widehat{-2\pi i x T} = (\hat{T})^{(1)}$;
4) $\check{\hat{T}} = T$.

Example: Fourier transform of the Delta function δ_ξ:

$$\widehat{\delta_\xi}(f) = \delta_\xi(\hat{f}) = \hat{f}(\xi) = T_{e^{-2\pi i x\xi}}(f).$$

Thus

$$\widehat{\delta_\xi} = T_{e^{-2\pi i x\xi}}$$

and the Fourier inversion formula gives

$$\widehat{T_{e^{2\pi i x\xi}}} = \delta_\xi.$$

Sometimes we just write $\widehat{\delta_\xi} = e^{-2\pi i x\xi}$ and $\widehat{e^{2\pi i x\xi}} = \delta_\xi$.

2.4.2. *Poisson summation formula and periodic distributions.* By Lemma 2.7, we know that

$$u := \sum_{k\in\mathbb{Z}} \delta_k,$$

defines a tempered distribution. From the definition, we know that the Poisson summation formula is equivalent to

Theorem 2.16. *The Fourier transform of u is equal to itself.*

Definition 2.17. *A distribution $T \in \mathcal{D}'$ is said to be* 1*-periodic if*

$$T(f(x)) = T(f_n), \ \ \forall\, f \in \mathcal{D}, \ n \in \mathbb{Z},$$

where $(f_n(x) := f(x+n)$.

It is clear that u is 1-periodic. In general, let $T \in \mathcal{D}'$ be 1-periodic, put

$$\phi_n(x) = \frac{\chi(x+n)}{\sum_{k\in\mathbb{Z}} \chi(x+n)},$$

where χ is defined in (12), then $\phi_0 \in \mathcal{D}_1$ and $\sum_{n\in\mathbb{Z}} \phi_n = 0$ gives

$$T(f) = \sum_{n\in\mathbb{Z}} T(f\phi_n) = \sum_{n\in\mathbb{Z}} T(f_n\phi_0), \ \ \forall\, f \in \mathcal{D}.$$

The above formula also implies that *every* 1*-periodic distribution is tempered.* Together with the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \hat{f}(x+n) = \sum_{n\in\mathbb{Z}} f(n) e^{-2\pi i n x}$$

it gives

$$T(\hat{f}) = \sum_{n\in\mathbb{Z}} T(e^{-2\pi i n x}\phi_0) f(n).$$

Thus we get

Theorem 2.18. *If $T \in \mathcal{D}'$ is* 1*-periodic then*

$$\hat{T} = \sum_{n\in\mathbb{Z}} T_{\mathbb{R}/\mathbb{Z}}(e^{-2\pi i n x})\delta_n,$$

where $T_{\mathbb{R}/\mathbb{Z}}(e^{-2\pi i n x}) := T(e^{-2\pi i n x}\phi_0)$ does not depend on ϕ_0.

Remark: In case $T = T_g$ for a 1-periodic continuous function g, we have

$$T(e^{-2\pi inx}\phi_0) = \int_0^1 g(x)e^{-2\pi inx}\, dx = \hat{g}(n)$$

Moreover, by the Fourier inversion formula, the above theorem gives

$$\int_{\mathbb{R}} f(x)g(x)\, dx = T(f) = \hat{T}(\widehat{f^-}) = \sum_{n\in\mathbb{Z}} \hat{g}(n) \int_{\mathbb{R}} f(x)e^{2\pi inx}\, dx,$$

which can be seen as a generalization of the Fourier series expansion $g(x) \sim \sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi inx}$.

2.4.3. *Elementary solution of $D + s$.* Recall that the Dirac operator D is defined by $-i\frac{d}{dx}$.

Definition 2.19. *Fix $\xi \in \mathbb{R}$, $s \in \mathbb{C}\backslash 2\pi\mathbb{Z}$. If a 1-periodic distribution T satisfies*

$$(D + s)T = \sum_{k\in\mathbb{Z}} \delta_{\xi+k},$$

then we call T an elementary solution of $D + s$ on $\mathbb{R}/\mathbb{Z}$.

Assume that T is an elementary solution. Apply the Fourier transform to the equation, Theorem 2.18 gives

$$(2\pi x + s)\hat{T} = \sum_{n\in\mathbb{Z}} e^{-2\pi in\xi}\delta_n.$$

Since $s \in \mathbb{C}\backslash 2\pi\mathbb{Z}$, we get

$$\hat{T} = \sum_{n\in\mathbb{Z}} \frac{e^{-2\pi in\xi}}{2\pi n + s}\delta_n.$$

Since $\check{\delta_n} = e^{2\pi inx}$, formally T is given by the following function

$$f(x) = \sum_{n\in\mathbb{Z}} \frac{e^{2\pi in(x-\xi)}}{2\pi n + s}.$$

Recall that we have proved that

$$\frac{\pi}{\sin \pi s}e^{i(\pi-x)s} = \sum_{n\in\mathbb{Z}} \frac{e^{inx}}{n + s}, \text{ if } x \in (0, 2\pi), \quad \pi \cot \pi s = \sum_{n\in\mathbb{Z}} \frac{1}{n + s}.$$

in Theorem 1.14. Thus we get

Theorem 2.20. *The elementary solution of $D+s$ on $\mathbb{R}/\mathbb{Z}$ is unique and is given by a piecewise smooth 1-period function.*

Remark: Assume that $\delta \in (0, 1)$. Another way to find the elementary solution is to find a piecewise smooth function on $[0, 1]$ such that

$$-if'(x) + sf(x) = \delta_\xi, \;\; f(0) = f(1),$$

where the first equation is defined in the sense of distribution. Let us rewrite the equation as

$$(fe^{isx})' = ie^{is\xi}\delta_\xi.$$

It is easy (try!) to check that the following *Heaviside function*

$$H(x) = 0, \;\; x < 0, \;\;\; H(x) = 1, \;\; x > 1, \;\; H(0) = \frac{1}{2}$$

satisfies $H' = \delta_0$. Thus we have

$$fe^{isx} = ie^{is\xi}H(x-\xi) + C.$$

Now $f(0) = f(1)$ gives

$$C = \frac{ie^{is\xi}}{e^{is}-1}.$$

Thus we get

$$f(x) = ie^{is(\xi-x)}\left(H(x-\xi) + \frac{1}{e^{is}-1}\right),$$

when $x \in [0,1]\backslash\{\xi\}$ and

$$f(\xi) = i\left(\frac{1}{2} + \frac{1}{e^{is}-1}\right) = \frac{1}{2}\cot(s/2).$$

2.4.4. *Fourier transform of the Heaviside function.* Recall that

$$2\pi ix\widehat{H-C} = \widehat{H'} = \hat{\delta}_0 = 1,$$

thus one might guess that

$$\widehat{H-C} = \frac{1}{2\pi i}\, p.v.\left(\frac{1}{x}\right)$$

and C should be $\frac{1}{2}$ since the right hand side is odd.

Theorem 2.21. $\widehat{H-\frac{1}{2}} = \frac{1}{2\pi i}\, p.v.\left(\frac{1}{x}\right)$.

Proof. By definition, for every $f \in \mathcal{S}$, we have

$$\widehat{H-\frac{1}{2}}(f) = \frac{1}{2}\left(\int_0^\infty \hat{f}(x)\,dx - \int_{-\infty}^0 \hat{f}(x)\,dx\right) = \frac{1}{2}\lim_{k\to\infty}\int_{\frac{1}{k}}^k \hat{f}(x) - \hat{f}(-x)\,dx.$$

Definition of $\hat{f}$ and the Euler formula give

$$\frac{\hat{f}(x)-\hat{f}(-x)}{2} = -\int_{\mathbb{R}} f(y)\sin(2\pi ixy)\,dy.$$

Thus

$$\widehat{H-\frac{1}{2}}(f) = \frac{1}{2\pi i}\lim_{k\to\infty}\int_{y\in\mathbb{R}} f(y)\frac{\cos(2\pi iky)-\cos(2\pi ik^{-1}y)}{y}\,dy,$$

Since $y^{-1}\cos ay$ is odd, we have

$$\int_{y\in\mathbb{R}} f(y)\frac{\cos(2\pi iky)-\cos(2\pi ik^{-1}y)}{y}\,dy = \int_0^\infty (f(y)-f(-y))\frac{\cos(2\pi iky)-\cos(2\pi ik^{-1}y)}{y}\,dy.$$

Since $f \in \mathcal{S}$, we have

$$\left|\int_N^\infty (f(y)-f(-y))\frac{\cos(2\pi iky)-\cos(2\pi ik^{-1}y)}{y}\,dy\right| \leqslant N^{-1},$$

when N is large enough. Moreover, the Riemann-Lebesgue lemma implies that

$$\lim_{k\to\infty}\int_0^N \frac{f(y)-f(-y)}{y}\cos(2\pi iky)\,dy = 0$$

since $\frac{f(y)-f(-y)}{y}$ is continuous on $[0, N]$. Thus

$$\widehat{H-\frac{1}{2}}(f) = \frac{1}{2\pi i}\lim_{N\to\infty}\lim_{k\to\infty}\int_0^N \frac{f(y)-f(-y)}{y}\cos(2\pi i k^{-1}y)\,dy = \frac{1}{2\pi i}\int_0^\infty \frac{f(y)-f(-y)}{y}\,dy.$$

Thus the theorem follows. □

2.5. **Fourier–Laplace transform and Paley–Wiener theorem.** Our starting point is the following basic observation: if $g \in \mathcal{D}_R$ then

$$\hat{g}(z) = \int_{\mathbb{R}} g(y)e^{-2\pi i y z}\,dy = \int_{-R}^{R} g(y)e^{-2\pi i y z}\,dy,$$

defines a holomorphic function on $z \in \mathbb{C}$, i.e. $\hat{g}$ is an *entire function*

Definition 2.22. *We call $\hat{g}(z), z \in \mathbb{C}$ the Fourier–Laplace transform of $g \in \mathcal{D}_R$.*

Let us write

$$z = a + ib, \qquad a, b \in \mathbb{R}$$

then

$$\hat{g}(a+ib) = \int_{-R}^{R} g(y)e^{-2\pi i a y}e^{2\pi b y}\,dy$$

gives

$$|\hat{g}(a+ib)| \leqslant e^{2\pi R|b|}\int_{-R}^{R}|g(y)|\,dy.$$

Moreover, for every natural number k, integration by parts gives

$$\widehat{g^{(k)}} = (2\pi i z)^k \hat{g}(z).$$

Thus

$$(2\pi)^k|z^k\hat{g}(z)| \leqslant e^{2\pi R|b|}\int_{-R}^{R}|g^{(k)}(y)|\,dy.$$

The smooth version of the Paley–Wiener theorem can be seen as an *inverse statement* of the above estimate.

Theorem 2.23 (Paley–Wiener–Schwartz)**.** *Let U be an entire function such that*

$$(1+|z|)^k\,|U(z)| \leqslant C_k\,e^{2\pi R\,|\mathrm{Im}\,z|}, \tag{13}$$

for every non-negative integer k. Then there exists $u \in \mathcal{D}_R$ whose Fourier–Laplace transform is U.

Proof. By the Fourier inversion formula, it suffices to define

$$u(x) = \int_{\mathbb{R}} U(y)e^{2\pi i x y}\,dy \tag{14}$$

and check that $u \in \mathcal{D}_R$. By (13), we know that $U|_{\mathbb{R}} \in \mathcal{S}$, thus $u \in \mathcal{S}$. Now it is enough to check that $u(x) = 0$ if $|x| > R$. The key of the proof is to use the Cauchy integral formula: notice that (13) permits us to shift the intrgration in (14) into the complex domains, which gives

$$u(x) = \int_{\mathbb{R}} U(y+ib)e^{2\pi i x(y+ib)}\,dy,$$

for every $b \in \mathbb{R}$. Estimating the integral by means of (13) with $k = 2$, we obtain

$$|u(x)| \leqslant C_2 e^{-2\pi x b + 2\pi R b}\int_{y\in\mathbb{R}}(1+|y|)^{-2}\,dy,$$

and the integral is convergent. If we choose $b = tx$ and let $t \to \infty$, it now follows that $u(x) = 0$ if $|x| > R$. □

In the next section, we shall study a distribution version of the above theorem.

2.5.1. *Fourier–Laplace transform of distributions with compact support.*

Definition 2.24. *ha*

Gives an L^2-version directly.

2.5.2. *A strong L^2-version of the Paley–Wiener theorem.* Follow Seip's notes (use exercise 3 in page 160 in [3]) to get the strong version.

Let us look at $L^2[-R, R]$ as the completion of $\mathcal{D}$ with respect to the following norm

$$||f||^2 := \int_{-R}^{R} |f(x)|^2 \, dx, \quad f \in \mathcal{D}.$$

Let $f = [\{f_n\}] \in L^2[-R, R]$, then

$$\hat{f}(z) := \lim_{n\to\infty} \int_{-R}^{R} f_n(x) e^{-2\pi i x z} \, dx,$$

defines an entire function.

Definition 2.25. *We call $\hat{f}(z), z \in \mathbb{C}$ the Fourier–Laplace transform of $f \in L^2[-R, R]$.*

Theorem 2.26 (Distribution version of the Paley–Wiener theorem)**.** *Let U be an holomorphic function on $\mathbb{C}$ such that*

$$(1 + |z|)^{-k} \, |U(z)| \leqslant C_k \, e^{2\pi R \, |\mathrm{Im}\, z|}, \tag{15}$$

for some integer k. Then there exists a unique continuous function, say g, on $\mathbb{R}$ such that

$$U(z) = \int_{\mathbb{R}} (2\pi i y)^k e^{-2\pi i z y} g(y) \, dy, \quad \forall \; z \in \mathbb{C},$$

and $g(x) = 0$ if $|x| > R$.

Proof. □

Let us start from the following smooth version of the

In this section, we shall study Fourier transform of k-th derivative of a continuous function with compact support.

Theorem 2.27. *Assume that $T = T_g^{(k)}$, where g is a continuous function whose support*

$$\mathrm{Supp}\, g := \{x \in \mathbb{R} : g(x) \neq 0\}$$

is bounded in $\mathbb{R}$. Then we can write $\hat{T} = T_h$, where h is a smooth function on $\mathbb{R}$ that extends to a holomorphic function on $\mathbb{C}$.

Proof. By definition, we have

$$\hat{T}(f) = (-1)^k T_g(\hat{f}^{(k)}) = \int_{\mathbb{R}^2} (2\pi i y)^k e^{-2\pi i x y} f(x) g(y) \, dx dy.$$

Thus it is enough to take

$$h(x) = \int_{\mathbb{R}} (2\pi i y)^k e^{-2\pi i x y} g(y) \, dy.$$

Since g has compact support and is continuous, we know that h is holomorphic. □

Definition 2.28. *We shall write $T_g^{(k)}$ as g^k and call the holomorphic function h in the above theorem the Fourier–Laplace transform $g^{(k)}$.*

Remark 1: In general, we call $T \in \mathcal{S}'$ a *distribution with compact support* if $T = T_g^{(k)}$ for a continuous function g with compact support. Usually we denote by $\mathcal{E}'$ the space of distributions with compact support. If $T \in \mathcal{E}'$ then we can write the *Fourier–Laplace transform* of T as

$$\hat{T}(z) := T(e^{-2\pi izx}), \ \ z \in \mathbb{C}.$$

Remark 2: In case $T = T_g$ for a continuous function g whose support is bounded in $[0, \infty]$, we call

$$\mathcal{L}(g)(s) := \int_0^\infty e^{-sy} g(y)\, dy = \hat{T}_g\left(\frac{s}{2\pi i}\right),$$

the *Laplace transform* of g.

Distribution version, and then the L^2–version

2.6. Sampling theorem.

2.7. Uncertainty principle.

2.8. Fast Fourier transform.

3. Wavelet analysis

Filter theory and signals, applications, TBA

4. Appendix 1: Definition of e, π and Euler's formula

4.1. Definition of e. Recall that: Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map (here linear map means $A(au + bv) = aA(u) + bA(v)$ for all a, b in $\mathbb{C}$ and all u, v in $\mathbb{C}^n$). We call $u \neq 0$ in $\mathbb{C}^n$ an *eigenvector* of A if

$$Au = \lambda u, \tag{16}$$

where λ is a constant in $\mathbb{C}$.

What is an eigenvector of the derivative ?

By (16), we want to find function $u : \mathbb{R} \to \mathbb{C}$ such that

$$u' = \lambda u.$$

Power series method: Assume that

$$u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

The following lemma gives:

$$u'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \cdots .$$

Lemma 4.1. $(x^n)' = nx^{n-1}, \ n = 1, 2, \cdots .$

Proof. If $n = 1$ then

$$x'(x) = \lim_{\triangle x \to 0} \frac{(x + \triangle x) - x}{\triangle x} = 1.$$

Assume the Lemma for $n = 1, \cdots, N-1$. Then $(fg)' = f'g + fg'$ gives

$$(x^N)' = (x^{N-1})' \cdot x + x^{N-1} \cdot x' = (N-1)x^{N-2} \cdot x + x^{N-1} = Nx^{N-1}.$$

The proof is complete. □

Exercise: Why we have $(fg)' = f'g + fg'$?

Now

$$u' = \lambda u \Leftrightarrow \lambda a_n = (n+1)a_{n+1},\ n = 0, 1, \cdots .$$

Thus

$$a_{n+1} = \frac{\lambda a_n}{(n+1)} = \frac{\lambda^2 a_{n-1}}{(n+1)n} = \cdots = \frac{\lambda^{n+1} a_0}{(n+1)n\cdots 1} = \frac{\lambda^{n+1} a_0}{(n+1)!},$$

where we define

$$n! = 1 \cdot 2 \cdots n.$$

Then we have

$$u(x) = u_0 \cdot (1 + \lambda x + \cdots + \frac{(\lambda x)^n}{n!} + \cdots).$$

Put

$$E(x) := 1 + x + \cdots + \frac{x^n}{n!} + \cdots .$$

Since for every $C > 0$,

$$\lim_{n\to\infty} \frac{C^n}{n!} = 0,$$

we know that $E(x)$ converges for all x in $\mathbb{C}$.

Theorem 4.2. *$E(\lambda x)$ is a unique solution of the eigenvalue equation*

$$u' = \lambda u,$$

with initail condition $u(0) = 1$.

Definition 4.3. *We shall define*

$$e := E(1) = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \cdots .$$

4.2. **Definition of the exponential function.** Let us write

$$e^2 = e \cdot e,\ e^3 = e^2 \cdot e,$$

and define e^m inductively by

$$e^{n+1} = e^n \cdot e.$$

Since e is positive, we can take the $q - th$ root of e^m, we write it as $e^{\frac{m}{q}}$. Thus for every $x \in \mathbb{Q}$, e^x is well defined. The following lemma tells us that $E(x)$ is an extension of e^x from $\mathbb{Q}$ to $\mathbb{C}$.

Lemma 4.4. *For every $x \in \mathbb{Q}$, we have $e^x = E(x)$.*

Proof. Since $E(1) = e$, it suffices to prove

$$E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2), \tag{17}$$

for every λ_1, λ_2 in $\mathbb{C}$. Notice that

$$(E(\lambda_1 x)E(\lambda_2 x))' = E(\lambda_1 x)' E(\lambda_2 x) + E(\lambda_2 x)' E(\lambda_1 x).$$

Put

$$G(x) = E(\lambda_1 x)E(\lambda_2 x).$$

Apply $E(\lambda x)' = \lambda E(\lambda x)$, we get

$$G' = (\lambda_1 + \lambda_2)G.$$

Notice that $G(0) = 1$. Thus Theorem 4.2 implies that

$$G(x) = E((\lambda_1 + \lambda_2)x).$$

Take $x = 1$, we get $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$. □

Exercise: Find a direct proof of $E(\lambda_1)E(\lambda_2) = E(\lambda_1 + \lambda_2)$ without using Theorem 4.2.

Definition 4.5. *We shall use the same symbol e^x to denote $E(x)$ for all x in $\mathbb{C}$ and call e^x the* ***exponential function****. If $x > 0$ then we define $\ln x$ as the unique real solution of $e^{\ln x} = x$.*

By Theorem 4.2, we know that e^x is fully determined by

$$(e^x)' = e^x, \ e^0 = 1.$$

4.3. **Definition of π and trigonometric functions.** : Fix $P_0 = (1, 0)$ in the unit circle

$$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

A counterclockwise rotation of P_0 gives a arc P_0P. The length, say $\theta(P)$, of the arc P_0P is a function of P. It is clear that the circumference diameter ratio is equal to $\theta(-1, 0)$.

Definition 4.6 (Definition of π). *We shall write the circumference diameter ratio as π.*

Denote by

$$F : \theta(P) \mapsto P,$$

the inverse function of $0 \leqslant \theta(P) \leqslant 2\pi$.

Definition 4.7. *We shall write $F(\theta) = (\cos\theta, \sin\theta)$.*

Notice that

$$F(0) = (1, 0) = F(2\pi), \ F(\pi) = (-1, 0), \ |F(\theta)| \equiv 1.$$

In particular, it gives

$$\sin(0) = \sin(2\pi) = 0, \ \cos(0) = \cos(2\pi) = 1.$$

By definition of θ, we have

$$\int_0^{\hat{\theta}} |F'(\theta)| \, d\theta = \hat{\theta}, \ 0 \leqslant \hat{\theta} \leqslant 2\pi,$$

which gives

$$|F'(\theta)| \equiv 1.$$

Now $F(\theta) \cdot F(\theta) \equiv 1$ implies

$$F' \cdot F + F \cdot F' = 2F \cdot F' \equiv 0.$$

Hence $F' \perp F$, thus we know that

$$F'(\theta) = (-\sin\theta, \cos\theta), \ \text{or } F'(\theta) = (\sin\theta, -\cos\theta).$$

But notice that $F'(0) = (0, 1)$, thus we must have

$$F'(\theta) = (-\sin\theta, \cos\theta),$$

which is equivalent to

$$(\cos\theta + i\sin\theta)' = i(\cos\theta + i\sin\theta).$$

Notice that $\cos 0 + i \sin 0 = 1$, thus Theorem 4.2 gives

Theorem 4.8 (Euler's formula). $e^{i\theta} = \cos\theta + i\sin\theta$.

Take $\theta = \pi$, we get the following Euler's identity

$$e^{i\pi} = -1.$$

Moreover, apply (17), we get

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)},$$

thus by Euler's formula, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2),$$

i.e.

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \tag{18}$$

and

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2. \tag{19}$$

5. Appendix 2: Lebesgue integral

TBA

6. Exercises

The following 12 exercise sets are NOT mandatory, but strongly recommended.

6.1. **Exercise set 1: Fejér kernel and its applications.** — From page 53–58, page 63 in [9], see also page 34–36 in [3].

6.1.1. *Fejér's theorem.*

6.1.2. *Weierstrass approximation theorem.*

6.1.3. *Poisson kernel.* see page 36 in [3] or page 54–58 in [9].

6.2. **Exercise set 2: Gibbs' Phenomenon.** — From page 94 in [9], see also page 44–46 in [3].

Let $f(x)$ denotes the sawtooth function defined by

$$f(x) = \frac{\pi - x}{2}$$

on the inteval $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to all of $\mathbb{R}$. The Fourier series of f is (try!)

$$f(x) \sim \frac{1}{2i} \sum_{|n|\neq 0} \frac{e^{inx}}{n} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

Exercise: Page 94, Exercise 19, 20 in [9].

6.3. **Exercise set 3: Some applications of Fourier series.** see Week 12, 15 other around the notes.

6.4. **Exercise set 4: Eigenfunctions of the Fourier transform.** page 97–101 in [3]. A canonical basis of $L^2(\mathbb{R})$. the Fourier transform.

REFERENCES

[1] A. Boggess and F. Narcowich, *A first course in Wavelets with Fourier Analysis*, Wiley, 2nd Edition, 2009.
[2] P. R. Chernoff, *Pointwise convergence of Fourier series*, The American Mathematical Monthly, **87** (1980), 399–400.
[3] H. Dym and H. P. McKean, *Fourier Series and Integrals*, Academic Press, 1972.
[4] F. Diamond and J. M. Shurman, *A first course in modular forms*, Vol. 228. New York: Springer, 2005.
[5] W. Feller, *An Introduction to Probability Theory and its Applications*, Volume I, 3rd edition (1968).
[6] G. W. Ford and G. E. Uhlenbeck, *Lectures in statistical mechanics*, 1974
[7] R. Gardner, *The Brunn–Minkowski inequality*, Bulletin of the American Mathematical Society 39.3 (2002): 355–405.
[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer.
[9] E. Stein and R. Shakarchi, *Fourier analysis*, Princeton lectures in analysis, 2003.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY
E-mail address: xu.wang@ntnu.no